

# Non-finitely generated relatively hyperbolic groups and Floyd quasiconvexity.

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August 16, 2011

## Abstract

The paper consists of two parts. In the first one we show that a relatively hyperbolic group  $G$  splits as a star graph of groups whose central vertex group is finitely generated and the other vertex groups are maximal parabolic subgroups. As a corollary we obtain that every group which admits 3-discontinuous and 2-cocompact action by homeomorphisms on a compactum is finitely generated with respect to a system of parabolic subgroups.

The second part essentially uses the methods of topological entourages developed in the first part. Using also Floyd metrics we obtain finer properties of finitely generated relatively hyperbolic groups. We show that there is a system of "tight" curves satisfying the property of horospherical quasiconvexity. We then prove that the Floyd quasigeodesics are tight and so the parabolic subgroups of  $G$  are quasiconvex with respect to the Floyd metrics. As a corollary we obtain that the preimage of a parabolic point by the Floyd map is the Floyd boundary of its stabilizer.

## 1 Introduction.

Let  $T$  be a compact Hausdorff space (compactum) containing at least 3 points. The action of a discrete group  $G$  by homeomorphisms on  $T$  is called *convergence action* if the induced action on the space  $\Theta^3 T$  of subsets of cardinality 3 is discontinuous. We say in this case that the action is *3-discontinuous*.

The action of  $G$  on  $T$  is called 2-cocompact if the induced action on the space  $\Theta^2 T$  of subsets of cardinality 2 is cocompact. An action is called *parabolic* if there is a unique fixed point, and *non-parabolic* in the opposite case.

Note that if  $G$  acts on  $T$  3-discontinuously and 2-cocompactly then the action is *geometrically finite*, i.e. every point of  $T$  is either conical or bounded parabolic [Ge1]. Conversely if a group  $G$  acts geometrically finitely on a **metrisable** space  $T$  then the action is 2-cocompact [Tu3]. In case when  $G$  is finitely generated then it is known from [Bo1] and [Ya] that the existence of a

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\*2010 Mathematics Subject Classification. Primary 20F65, 20F67 - Secondary 30F40, 57M07, 22D05

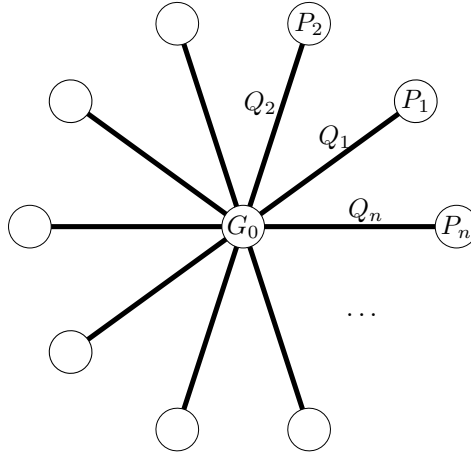
Key words: Relatively hyperbolic groups, Relatively finitely generated groups, Floyd quasiconvexity.

geometrically finite action of  $G$  on a metrisable space  $T$  is equivalent to the “classical” Farb’s definition of relative hyperbolicity [Fa]. These facts justify the following "dynamical" definition.

**Convention.** A group  $G$  is *relatively hyperbolic* if it admits a non-parabolic 3-discontinuous and 2-cocompact action on a compactum  $T$ .

Our first result shows that a non-finitely generated relatively hyperbolic group can be "nicely" approximated by a finitely generated one.

**Theorem A.** *Let  $G$  be a relatively hyperbolic group with respect to a collection of parabolic subgroups  $\{P_1, \dots, P_n\}$ . Then there exists a finitely generated subgroup  $G_0$  of  $G$  which is relatively hyperbolic with respect to the collection  $\{Q_i = P_i \cap G_0 \mid i = 1, \dots, n\}$  such that  $G$  is the fundamental group of the star graph*



whose central vertex group is  $G_0$  and all other vertex groups are  $P_i$  ( $i = 1, \dots, n$ ).

Furthermore for every finite subset  $K \subset G$  the subgroup  $G_0$  can be chosen to contain  $K$ .

Theorem A admits several useful applications consisting in generalizing results and methods known for finitely generated groups to the case of non-finitely generated groups. We used it in [GePo3, Proposition 7.1.2] to show that the existence of a 3-discontinuous and 2-cocompact action of a group  $G$  on a compactum implies that there is a hyperbolic  $G$ -cofinite graph which is *fine*. This generalizes A. Yaman’s theorem [Ya] for non-finitely generated (and even uncountable) relatively hyperbolic groups.

To describe another application let us fix a system  $\mathcal{H}$  of subgroups of  $G$ . Then  $G$  is called *finitely generated with respect to  $\mathcal{H}$*  if  $G$  is generated by  $S \cup \mathcal{H}$  where  $S \subset G$  is a finite subset.

**Corollary** (Corollaries 3.39, 3.40). *Let  $G$  be a group acting 3-discontinuously and 2-cocompactly on a compactum  $T$ . Then  $G$  is finitely generated with respect to the stabilizers of the parabolic points.*

Furthermore if the above action of  $G$  on  $T$  is without parabolics then  $G$  is finitely generated.

Note that the above Corollary was used in [GePo1, Appendix] to obtain a short proof of Bowditch's theorem that a 3-discontinuous and 3-cocompact action of a group implies its hyperbolicity.

We emphasize that only few known until now results about relatively hyperbolic groups dealt with non-finitely generated groups. In particular according to Osin's definition such groups should satisfy the above Corollary [Os, Definition 2.35]. The equivalence between several known definitions was proved by C. Hruska [Hr]. However the essential assumptions of his theorem are: the group  $G$  is at most countable, and the compactum  $T$  must be metrisable. We do not impose either of them in Theorem A.

The proof of Theorem A is based on the theory of topological entourages on the space  $\mathbf{S}^2T$  of non-ordered pairs of points of  $T$ . In Section 3 using a discrete system  $A$  of entourages on  $T$  we construct a graph of entourages  $\mathcal{G}$  on which  $G$  acts cocompactly. The subgroup  $G_0$  will be chosen as the stabilizer of a connected component of a refined graph  $\tilde{\mathcal{G}}$  having the same set of vertices:  $\tilde{\mathcal{G}}^0 = \mathcal{G}^0 = A$ . We will extensively use a system of tubes and horospheres on the graph  $\mathcal{G}$  to establish the existence of the splitting of  $G$  as a graph of groups.

The second part of the paper deals with finitely generated relatively hyperbolic groups. It is based on the topological methods developed in the first one. Starting with Section 4 we use the Floyd completion of locally finite graphs and restrict ourselves to the case of *finitely generated* relatively hyperbolic groups. Let  $\Gamma$  be an abstract locally finite, connected graph admitting a cocompact and discontinuous action of a finitely generated group  $G$  (e.g. its Cayley graph or the graph of entourages  $\mathcal{G}$ ). According to W. Floyd by rescaling the graph distance  $d$  of  $\Gamma$  by a scalar function  $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  one obtains the Cauchy completion  $\bar{\Gamma}_f$  of the metric space  $(\Gamma, \delta_f)$  where  $\delta_f$  is the rescaled metric. We call this space *Floyd completion* (see Section 4 below). The action of  $G$  extends continuously to  $\bar{\Gamma}_f$ . By [Ge2] there exists an equivariant continuous map  $F$  between the *Floyd boundary*  $\partial_f \Gamma = \bar{\Gamma}_f \setminus \Gamma$  and the space  $T$ . The kernel of the map  $F$  was described in [GePo1, Thm A]. Namely if it is not a single point then it is equal to the topological boundary  $\partial(\text{Stab}_{Gp})$  of the stabilizer  $\text{Stab}_{Gp}$  of a parabolic point  $p \in T$ . We denote by  $\partial_f \text{Stab}_{Gp}$  the Floyd boundary of  $\text{Stab}_{Gp}$  corresponding to a function  $f$ .

A subset  $X$  of  $\Gamma$  is called *Floyd ( $r$ -)quasiconvex* if every Floyd geodesic (with respect to the metric  $\delta_f$ ) with the endpoints in  $X$  belongs to  $r$ -neighborhood  $N_r(X)$  for the graph metric  $d$  and some  $r > 0$ . In particular if the scalar function  $f$  is the identity the Floyd quasiconvexity coincides with the standard one. It is known (see e.g. [GePo1, Corollary 3.9]) that the parabolic subgroups are quasiconvex with respect to  $d$ . Our next Theorem establishes the Floyd quasiconvexity of the parabolic subgroups.

**Theorem C.** *Let  $G$  be a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum  $T$ . Let  $\Gamma$  be a locally finite, connected graph admitting a cocompact discontinuous action of  $G$ . Then there exists a Floyd scaling function  $f$ , such that every parabolic subgroup  $H$  of  $G$  is Floyd quasiconvex for the Floyd metric  $\delta_f$ .  $\square$*

As a consequence of Theorem C we obtain the following Corollary which answers our question [GePo1, 1.1]:

**Corollary 7.8** *For a scaling function  $f$  satisfying conditions (1 – 3) (see Section (7)) one has*

$$F^{-1}(p) = \partial_f(\text{Stab}_{Gp})$$

for every parabolic point  $p \in T$ . □

We note that Corollary 7.8 provides a complete generalization of the Floyd theorem [F] to the case of relatively hyperbolic groups. W. Yang used Corollary 7.8 in his thesis [Yang] to show that the well-known conjecture that the non-triviality of the Floyd boundary of a finitely generated group  $G$  implies the relative hyperbolicity of  $G$  is equivalent to the fact that every relatively hyperbolic group with respect to a system of non-relatively hyperbolic subgroups acts geometrically finitely on its Floyd boundary.

The proof of Theorem C (and Corollary 7.8) is given in Section 7 and is based on a description of a family of curves which are quasigeodesics locally everywhere and which are not horocycles (see Definition 6.1). Their properties are described in the following Theorem (see Section 6 for more details):

**Theorem B.** *For every tight curve  $\gamma$  in the graph of entourages  $\mathcal{G}$  there exists a quasigeodesic  $\alpha \subset A$  such that every non-horospherical vertex of  $\gamma$  belongs to a uniform neighborhood of  $\alpha$ .*

The main step in proving Theorem C is to show that every Floyd quasigeodesic is tight. We notice that the graph of entourages  $\mathcal{G}$  plays here a special role and in the proofs of Theorems B and C we deal mainly with it.

This is our second paper in a series of papers about relatively hyperbolic groups. Keeping the same definition of the relative hyperbolicity here we apply however different methods based on the theory of discrete systems of entourages not used in [GePo1].

**Acknowledgements.** During the work on this paper both authors were partially supported by the ANR grant BLAN 07 – 2183619. We are grateful to the Max-Planck Institute für Mathematik in Bonn, where a part of the work was done. We also thank the Brazilian-French cooperation grant having supported our work.

The authors are thankful to Wenyan Yang for very useful remarks and corrections.

## 2 Convergence Groups.

By *compactum* we mean a compact Hausdorff space. Let  $\mathbf{S}^n T$  denote the quotient of the product space  $\underbrace{T \times \dots \times T}_n$  by the action of the permutation group on  $n$  symbols. The elements of  $\mathbf{S}^n T$  are generalized non-ordered  $n$ -tuples (i.e. an element may belong to a tuple with some multiplicity). Let  $\Theta^n T$  be the subset of  $\mathbf{S}^n T$  whose elements are non-ordered  $n$ -tuples with all distinct components. Put  $\Delta^n T = \mathbf{S}^n T \setminus \Theta^n T$ , the set  $\Delta^2 T$  is just the diagonal of  $T^2$ .

**Convention.** *If the opposite is not stated all group actions on compacta are assumed to have the convergence property.*

We refer to [Bo2], [GePo1], [GM], [Fr], [Tu2] where standard facts related to the convergence groups are proved. We recall below few facts really used in this paper.

The limit set  $\Lambda(G)$  is the set of accumulation (limit) points of the  $G$ -orbit for the action of  $G$  on  $T$ . It is known that either  $|\Lambda(G)| \in \{0, 1, 2\}$  in which case the action  $G \curvearrowright T$  is called *elementary* or it is a perfect set and the action is not elementary [Tu2].

An elementary action of a group on  $T$  is called *parabolic* if there is unique fixed point called parabolic fixed point.

A limit point  $x \in \Lambda(G)$  is called *conical* if there exists an infinite sequence  $g_n \in G$  and distinct points  $a, b \in T$  such that

$$\forall y \in T \setminus \{x\} : g_n(y) \rightarrow a \in T \wedge g_n(x) \rightarrow b.$$

A parabolic fixed point  $p \in \Lambda(G)$  is called *bounded parabolic* if the quotient space  $(\Lambda(G) \setminus \{p\})/\text{Stab}_G p$  is compact.

A set  $M$  is called *G-finite* if  $M/G$  is a finite set.

An action of a group  $G$  on a compactum  $T$  is called *geometrically finite* if every limit point of  $T$  is either conical or bounded parabolic. As we have pointed out in the Introduction if  $G \curvearrowright T$  is a 3-discontinuous and 2-cocompact action then it is also a geometrically finite one. The opposite statement is also true if one assumes that  $T$  is metrizable.

**Notation.** From now on we fix the notation  $\mathcal{P}$  for the set of parabolic points for the 3-discontinuous and 2-cocompact action  $G \curvearrowright T$ .

### 3 Exhaustion of non-finitely generated relatively hyperbolic groups by finitely generated ones.

#### 3.1 Entourages, shadows, betweenness relation.

The following definition is motivated by [Bourb] and [W].

**Definition 3.1.** Let  $T$  be a compactum. Any neighborhood of the diagonal  $\Delta^2 T$  in  $\mathbf{S}^2 T$  is called *entourage* of  $T$ . The set of all entourages of  $T$  is denoted by  $\text{Ent } T$ .

**Convention.** By definition an entourage consists of **non-ordered** pairs. However sometimes we identify an entourage  $\mathbf{e} \in \text{Ent } T$  with the **symmetric** neighborhood  $\tilde{\mathbf{e}}$  of the diagonal in  $T \times T$ .

We denote the entourages by bold small characters.

An entourage  $\mathbf{e}$  determines a graph whose vertex set is  $T$ , and two vertices  $x, y$  are joined by an edge if and only if  $\{x, y\} \in \mathbf{e}$ . Denote by  $\Delta_{\mathbf{e}}$  the corresponding graph distance which is the maximal distance function with the property  $\{x, y\} \in \mathbf{e} \implies \Delta_{\mathbf{e}}(x, y) \leq 1$ . Note that  $\Delta_{\mathbf{e}}(x, y) = \infty$  if and only if  $x$  and  $y$  belong to different connected components of the graph. A set  $U \subset T$  is called **e-small** if its **e-diameter** is at most 1.

The set of all **e-small** sets is denoted by  $\text{Small}(\mathbf{e})$ . For subsets  $a, b \subset T$  we define  $\Delta_{\mathbf{e}}(a, b) = \inf\{\Delta_{\mathbf{e}}(x, y) \mid x \in a, y \in b\}$  and  $\widetilde{\Delta}_{\mathbf{e}}(a, b) = \sup\{\Delta_{\mathbf{e}}(x, y) \mid x \in a, y \in b\}$ . From the triangle inequality we have the inequality  $\Delta_{\mathbf{e}}(a, b) \geq \Delta_{\mathbf{e}}(a, c) - \widetilde{\Delta}_{\mathbf{e}}(c, b)$  frequently used further.

For a subset  $a \subset T$  define its **e-neighborhood**  $a\mathbf{e}$  as  $\{x \in T \mid \Delta_{\mathbf{e}}(x, a) \leq 1\}$ .

For a subset  $o$  of  $T$  its "*convex hull*" in  $T \sqcup \text{Ent } T$  is the set

$$\tilde{o} = o \cup \{\mathbf{e} \in \text{Ent } T : o' \in \text{Small}(\mathbf{e})\}. \quad (\dagger)$$

We equip the space  $T \sqcup \text{Ent } T$  with the topology generated by the "*convex hulls*" of open subsets of  $T$  and the single-point subsets of  $\text{Ent } T$ . Namely a set  $w$  in  $T \sqcup \text{Ent } T$  is declared *open* if for every point  $t \in w \cap T$  there exists  $o \in \text{Open } T$  such that  $t \in o$  and  $\tilde{o} \subset w$ . In particular  $\text{Ent } T$  is a discrete open subset and  $T$  is a closed subspace of  $T \sqcup \text{Ent } T$ .

**Remark.** The definition of the topology on  $T \sqcup \text{Ent } T$  is motivated by the topology on the compactified real hyperbolic space  $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$  given by open sets  $o \subset \partial_\infty \mathbb{H}^n$  and their convex hulls in  $\mathbb{H}^n$ . This analogy can be explained as follows. A bounded subset  $B \subset \mathbb{H}^n$  defines an entourage  $\mathbf{e}_B \in \text{Ent}(\partial \mathbb{H}^n)$  in the following way:  $\{x, y\} \in \mathbf{e}_B$  if and only if the geodesic  $\gamma(x, y)$  with the endpoints  $x$  and  $y$  misses  $B$ . The entourage  $\mathbf{e}_B$  is close to a point  $a \in \partial \mathbb{H}^n$  if for an open neighborhood  $o \subset \partial \mathbb{H}^n$  the convex hull  $\tilde{o}$  of  $o$  in  $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$  contains  $B$  (see Figure 1).  $\square$

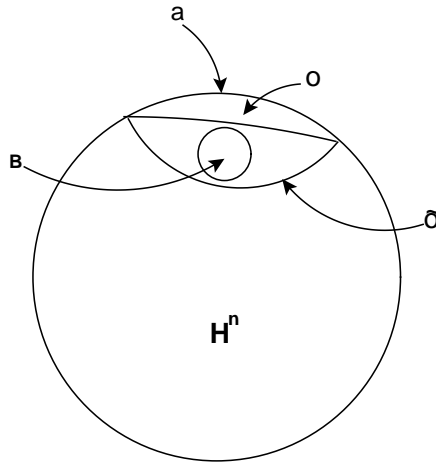


Figure 1: Bounded set in  $\mathbb{H}^n$  and its visibility entourage.

**Definition 3.2.** [Ge1] Two entourages  $\mathbf{a}$  and  $\mathbf{b}$  are said to be *unlinked* if there exist  $a \in \text{Small}(\mathbf{a})$  and  $b \in \text{Small}(\mathbf{b})$  such that  $T = a \cup b$ . We denote this relation by  $\mathbf{a} \bowtie \mathbf{b}$ . In the opposite case we say that  $\mathbf{a}$  and  $\mathbf{b}$  are *linked*, and write  $\mathbf{a} \# \mathbf{b}$ .  $\square$

Denote by  $\mathbf{La}$  the set  $\{\mathbf{b} \in \text{Ent } T \mid \mathbf{a} \# \mathbf{b}\}$ . It is enough for our purposes to consider only sufficiently small entourages implying the following.

**Convention.** All considered entourages are supposed to be self-linked :

$$\mathbf{a} \in \text{Ent } T : \mathbf{a} \# \mathbf{a}. \quad (1)$$

**Definition 3.3.** [Ge1] Let  $\mathbf{a}$  and  $\mathbf{b}$  be two unlinked entourages. We define the following "shadow" sets :

$$\text{Sh}_{\mathbf{a}} \mathbf{b} = \{a \in \text{Small}(\mathbf{a}) \mid a' \in \text{Small}(\mathbf{b})\},$$

and

$$\text{sh}_a \mathbf{b} = \bigcap \text{Sh}_a \mathbf{b} = (\bigcup \text{Sh}_b \mathbf{a})'.$$

It is shown in [Ge1, Lemma S0] that if  $\mathbf{a} \bowtie \mathbf{b}$  and  $\text{diam}_a T > 2$  then  $\text{sh}_a \mathbf{b} \neq \emptyset$ ; and if  $\text{diam}_a T > 4$  then  $\text{sh}_a \mathbf{b}$  has a nonempty interior.

**Convention.** We consider only the entourages  $\mathbf{a}$  with  $\text{Diam}_a T > 4$ . So every shadow has non-empty interior.

**Remark.** In the hyperbolic space  $\mathbb{H}^n$  the shadows give rise to the sets at the sphere at infinity illustrated on the Figure 2.

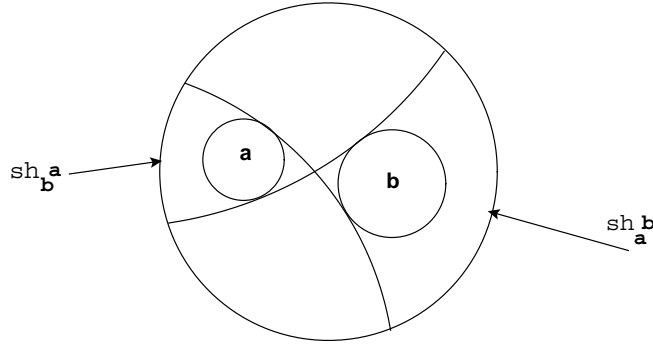


Figure 2: Shadows  $\text{sh}_b \mathbf{a}$  and  $\text{sh}_a \mathbf{b}$ .

**Definition 3.4.** (*Betweenness relation*). Let  $k$  be a positive integer.

- 1) Suppose  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \text{Ent } T$ . We say that an entourage  $\mathbf{b}$  lies *between* (or *k-between*)  $\mathbf{a}$  and  $\mathbf{c}$ , and write  $\mathbf{a} - \mathbf{b} - \mathbf{c} (k)$  (or simply  $\mathbf{a} - \mathbf{b} - \mathbf{c}$ ), if  $\mathbf{a} \bowtie \mathbf{b} \bowtie \mathbf{c}$  and  $\Delta_b(\text{sh}_b \mathbf{a}, \text{sh}_b \mathbf{c}) > k$ .
- 2) Let  $\mathbf{a}, \mathbf{b} \in \text{Ent } T$  and let  $p \in T$ . We say that  $\mathbf{b}$  lies *between* (or *k-between*)  $\mathbf{a}$  and  $p$  if  $\mathbf{a} \bowtie \mathbf{b}$  and  $\Delta_b(\text{sh}_b \mathbf{a}, b) > k$  for any  $\mathbf{b}$ -small neighborhood  $b$  of  $p$ . We write  $\mathbf{a} - \mathbf{b} - p (k)$  (or simply  $\mathbf{a} - \mathbf{b} - p$ ) in this case.
- 3) Let  $\mathbf{b} \in \text{Ent } T$  and let  $p, q \in T$  be two distinct points. We say that  $\mathbf{b}$  lies *between* (or *k-between*)  $p$  and  $q$ , and write  $q - \mathbf{b} - p (k)$  (or simply  $q - \mathbf{b} - p$ ), if  $\Delta_b(b_1, b_2) > k$  for any  $\mathbf{b}$ -small neighborhoods  $b_1$  and  $b_2$  of the points  $p$  and  $q$  respectively.

**Remarks 3.5.** a) The betweenness relations 2) and 3) represent an extension "by continuity" of the relation 1) between entourages to the points of  $T$ . Note that the middle object in the relation  $\mathbf{a} - \mathbf{b} - \mathbf{c}$  is always an entourage.

Note also that if  $\Delta_{\mathbf{b}}(\text{sh}_{\mathbf{b}}\mathbf{a}, b_0) > k$  for some  $\mathbf{b}$ -small neighborhood  $b_0$  of  $p$  then for any such  $b$  we have  $\Delta_{\mathbf{b}}(\text{sh}_{\mathbf{b}}\mathbf{a}, b) \geq \Delta_{\mathbf{b}}(\text{sh}_{\mathbf{b}}\mathbf{a}, b_0) - \Delta_{\mathbf{b}}(b, b_0) > k - 2$  as  $p \in b \cap b_0$  and  $\tilde{\Delta}_{\mathbf{b}}(b, b_0) \leq 2$ . Therefore we will always assume further that  $k > 2$ .

b) Definition 3.4 in cases 2) and 3) differs from the corresponding definition in [Ge1] where the condition  $\Delta_{\mathbf{b}}(\text{sh}_{\mathbf{a}}\mathbf{b}, p) > k$  is stated instead of 2). The above betweenness definition is stronger than that of [Ge1] and so is easier to use. However both of them are quite close: the  $k$ -betweenness 2) implies  $k$ -betweenness of [Ge1]. From the other hand since the diameter of any small neighborhood is less than 1 the  $k + 1$ -betweenness of [Ge1] implies (by the triangle inequality) the  $k$ -betweenness 2). We will use results of [Ge1] keeping in mind this relation.

**Lemma 3.6.** (*Continuity property*). Suppose that  $\mathbf{a} - \mathbf{c} - p(k)$  ( $k \in \mathbb{N}$ ) where  $\mathbf{a} \in T \sqcup \text{Ent } T$ ,  $\mathbf{c} \in \text{Ent } T$ ,  $p \in T$ . Suppose that  $p \in T$  is an accumulation point for an infinite subset  $B$  of  $\text{Ent } T$ . Then there exists  $\mathbf{b} \in B$  such that  $\mathbf{a} - \mathbf{c} - \mathbf{b}(k)$ .

*Proof:* Let first  $\mathbf{a} \in \text{Ent } T$  be an entourage. Let  $c = U_p$  be an open  $\mathbf{c}$ -small set containing  $p$  such that  $\Delta_{\mathbf{c}}(c, \text{sh}_{\mathbf{c}}\mathbf{a}) > k$ . By definition of the topology of  $T \sqcup \text{Ent } T$  the complement  $c'$  is  $\mathbf{b}$ -small for some  $\mathbf{b} \in B$ . Then  $c' \subset \bigcup \text{Sh}_{\mathbf{b}}\mathbf{c}$ , and  $c \supset \text{sh}_{\mathbf{c}}\mathbf{b} = (\bigcup \text{Sh}_{\mathbf{b}}\mathbf{c})'$ . Thus  $\Delta_{\mathbf{c}}(\text{sh}_{\mathbf{c}}\mathbf{a}, \text{sh}_{\mathbf{c}}\mathbf{b}) > \Delta_{\mathbf{c}}(\text{sh}_{\mathbf{c}}\mathbf{a}, c) > k$ .

If now  $a \in T$  then for a  $\mathbf{c}$ -small neighborhood  $U$  containing  $a$ , we obtain similarly  $\Delta_{\mathbf{c}}(U, \text{sh}_{\mathbf{c}}\mathbf{b}) > \Delta_{\mathbf{c}}(U, U_p) > k$ . So we still have  $a - \mathbf{c} - \mathbf{b}(k)$  for  $\mathbf{b} \in B$ .  $\square$

**Definition 3.7.** (*Tubes*). [Ge1] A sequence  $P$  of elements  $\mathbf{a}_n$  of  $T \sqcup \text{Ent } T$  is called  $k$ -tube (or tube) if

$$\forall n : (\mathbf{a}_n \bowtie \mathbf{a}_{n+1}) \wedge (\mathbf{a}_{n-1} - \mathbf{a}_n - \mathbf{a}_{n+1}(k))$$

whenever  $\mathbf{a}_{n \pm 1}$  are defined.  $\square$

**Lemma 3.8.** 1) (*Ordering*) For any three entourages at most one can be between the others.

2) (*Convexity*) If  $\mathbf{a} - \mathbf{b} - \mathbf{c}(4)$  and  $\mathbf{a}, \mathbf{c} \in \text{Ld}$  then  $\mathbf{b} \in \text{Ld}$ .

*Proof:* 1) Indeed if not, we obtain  $\mathbf{a} - \mathbf{b} - \mathbf{c}$  and  $\mathbf{a} - \mathbf{c} - \mathbf{b}$  for some  $\mathbf{a}, \mathbf{b}, \mathbf{c}$ . The transitivity of the betweenness relation [Ge1] would imply  $\mathbf{a} - \mathbf{b} - \mathbf{a}$  and so  $\mathbf{a} \bowtie \mathbf{a}$  which is impossible by our convention (1).

2) Otherwise  $\mathbf{b} \bowtie \mathbf{d}$  and we have  $T = b \cup d = a \cup b_1 = c \cup b_2$  where  $b_i, b \in \text{Small}(\mathbf{b})$  ( $i = 1, 2$ ),  $d \in \text{Small}(\mathbf{d})$ ,  $a \in \text{Small}(\mathbf{a})$ ,  $c \in \text{Small}(\mathbf{c})$ . It follows that  $b \cap b_1 = \emptyset$  or  $b \cap b_2 = \emptyset$  since otherwise  $\Delta_{\mathbf{b}}(b_1, b_2) \leq 2$  and we would have  $\Delta_{\mathbf{b}}(\text{sh}_{\mathbf{b}}\mathbf{a}, \text{sh}_{\mathbf{b}}\mathbf{c}) \leq 2 + \tilde{\Delta}_{\mathbf{b}}(\text{sh}_{\mathbf{b}}\mathbf{a}, b_1) + \tilde{\Delta}_{\mathbf{b}}(\text{sh}_{\mathbf{b}}\mathbf{c}, b_2) \leq 4$  (as  $\text{sh}_{\mathbf{b}}\mathbf{a} \subset b_1$  and  $\text{sh}_{\mathbf{b}}\mathbf{c} \subset b_2$ ) which is impossible. If, for instance,  $b \cap b_1 = \emptyset$  then  $b_1 \subset d$  and  $\mathbf{a} \bowtie \mathbf{d}$ . A contradiction.  $\square$



### 3.2 Discrete sets of entourages. Horospheres.

Until the end of Section 3 we fix a 3-discontinuous 2-cocompact action  $G \curvearrowright T$  of a group  $G$  on a compactum  $T$ .

**Definition 3.9.** A set  $A$  of entourages on  $T$  is called *discrete* if

$$\forall \mathbf{w} \in \text{Ent } T : |\{\mathbf{a} \in A : \mathbf{a} \# \mathbf{w}\}| < \infty. \quad (1)$$

By [Ge1, Proposition P] the set  $\{g \in G : g\mathbf{a} \# \mathbf{w}\}$  is finite for all  $\mathbf{w}, \mathbf{a} \in \text{Ent } T$ . This property is called *Dynkin property* [Fu]. Hence every  $G$ -finite set is discrete.

Let  $A \subset \text{Ent } T$  be a  $G$ -finite set of entourages. Denote by  $\tilde{T}$  the subspace  $T \sqcup A$  of  $T \sqcup \text{Ent } T$ . Since  $A$  is discrete  $\tilde{T}$  is compact [Ge1, Proposition D].

**Definition 3.10.** Let  $\mathcal{G} = \mathcal{G}_A$  be the graph whose vertex set  $\mathcal{G}^0$  is  $A$  and the edge set  $\mathcal{G}^1$  is the set of pairs  $\{\mathbf{a}, \mathbf{b}\}$  such that  $\mathbf{a} \# \mathbf{b}$ . Denote by  $d_A$  the corresponding graph distance.

Since  $G$  acts on  $T$  by homeomorphisms it acts isometrically on  $(\mathcal{G}, d_A)$ .

**Lemma 3.11.** *The group  $G$  is finitely generated if and only if there exists a connected graph  $\mathcal{G}_A$ .*

*Proof:* Suppose first that  $G$  admits a finite set of generators  $S$  ( $\text{id} \in S$ ). Since  $A$  is  $G$ -finite we have  $A = \bigcup_{i=1}^l G(\mathbf{a}_i)$ . Any entourage  $\mathbf{a}_i$  contains a sub-entourage  $\mathbf{a}'_i$  such that

$$\forall s \in S : \mathbf{a}'_i \# s\mathbf{a}'_j \quad (i, j \in \{1, \dots, l\}).$$

So up to choosing the entourages  $\mathbf{a}_i$  ( $i = 1, \dots, l$ ) to be sufficiently small we can assume that the above property is satisfied. Then all vertices in the set  $\bigcup_i S\mathbf{a}_i$  are pairwise connected by edges. For any vertex  $\mathbf{v} \in \mathcal{G}_A$  there exists  $i \in \{1, \dots, l\}$  and  $g \in G$  such that  $\mathbf{v} = g(\mathbf{a}_i)$  and  $g = s_{i_1}s_{i_2}\dots s_{i_k}$  ( $s_{i_j} \in S$ ). Then  $\mathcal{G}_A$  contains the edges  $e = (s_{i_k}(\mathbf{a}_i), \mathbf{a}_i)$ ,  $e' = (s_{i_{k-1}}(\mathbf{a}_i), \mathbf{a}_i)$ , and so the path  $s_{i_{k-1}}e \cup e'$  between  $\mathbf{a}_i$  and  $s_{i_{k-1}}s_{i_k}(\mathbf{a}_i)$ . Continuing in this way we obtain a path between  $\mathbf{v}$  and  $\mathbf{a}_i$ .

Conversely suppose that  $\mathcal{G}_A$  is connected. Let  $S$  be the set  $\{s \in G \mid s\mathbf{a}_j \# \mathbf{a}_i\}$  where  $A = \bigcup_{i=1}^l G\mathbf{a}_i$ . By Dynkin property the set  $S$  is finite. For any  $g \in G$  there is a path  $l = \{\mathbf{a}_i, \mathbf{b}_2, \dots, \mathbf{b}_{n-1}, \mathbf{a}\} \subset \mathcal{G}_A$  between the vertices  $\mathbf{a} = g(\mathbf{a}_i)$  and  $\mathbf{a}_i$ . Then  $\mathbf{b}_2 \# \mathbf{a}_i$  so  $\exists s_1 \in S : \mathbf{b}_2 = s_1(\mathbf{a}_i)$ . Thus  $s_1^{-1}\mathbf{b}_2 \# \mathbf{a}_i$  and  $\exists s_2 \in S : \mathbf{b}_3 = s_1s_2\mathbf{a}_i$ . Continuing in this way we obtain  $\mathbf{a} = s_1s_2\dots s_n\mathbf{a}_i$ . Then  $g^{-1}(s_1s_2\dots s_n)$  fixes  $\mathbf{a}_i$  and so belongs to  $S$  (by (1) of 3.1). The Lemma is proved.  $\square$

It follows from Dynkin property and our convention (1) that the stabilizer of each edge and each vertex of  $\mathcal{G}$  is finite. The action  $G \curvearrowright T$  is 2-cocompact so by [Ge1, Proposition E] we can suppose that the set  $A$  is a single orbit  $G(\mathbf{a}_0)$  ( $\mathbf{a}_0 \in \text{Ent } T$ ) having the following properties :

i) *m-separation property*:

$$\forall (p, q) \in \Theta^2 T \exists \mathbf{a} \in A : p - \mathbf{a} - q(m), \quad (2)$$

for a fixed  $m \in \mathbb{N}$ .

ii) *generating property*:

$$\forall \mathbf{u} \in \text{Ent } T \exists \mathbf{a}_i \in A \ (i = 1, \dots, l) : \mathbf{u} \supset \bigcap_{i=1}^l \mathbf{a}_i. \quad (3)$$

i.e.  $A$  generates  $\text{Ent } T$  as a filter.

**Convention 3.12.** From now on we fix an unlinked entourage  $\mathbf{a}_0 \in \text{Ent } T$  (see (1) of 3.1) such that its orbit  $A = G(\mathbf{a}_0)$  satisfies  $m$ -separating and generating properties. The value of  $m$  can be easily restored in each statement. Keeping in mind that this value might be needed to be increased further we just suppose that  $m$  is sufficiently large.

Furthermore if  $G$  is finitely generated we will always assume (by Lemma 3.11) that the graph  $\mathcal{G}$  is connected.

**Remarks.** The graph  $\mathcal{G}$  plays the role of the Cayley graph  $\mathcal{Ca}(G)$  if  $G$  is finitely generated, however by Dynkin property it is always a locally finite graph. The space  $\tilde{T} = T \sqcup A$  is a compactification of  $A = \mathcal{G}^0$  similar to the Floyd completion (see Section 4). Every action  $G \curvearrowright T$  can be naturally extended to the space  $\tilde{T}$ .

**Lemma 3.13.** *The space  $\tilde{T} = T \sqcup A$  is a compactum.*

*Proof:* The space  $T$  is Hausdorff. To prove that  $\tilde{T}$  is Hausdorff we will consider three different cases. Let first  $x, y$  be distinct points of  $T$  then there exist disjoint closed neighborhoods  $U_x$  and  $U_y$  in  $T$ . Their convex hulls  $\tilde{U}_x = U_x \cup \{\mathbf{e} \in A : U'_x \in \text{Small}(\mathbf{e})\}$  and  $\tilde{U}_y = U_y \cup \{\mathbf{d} \in A : U'_y \in \text{Small}(\mathbf{d})\}$  are neighborhoods of these points in the topology of  $\tilde{T}$  induced from  $T \sqcup \text{Ent } T$  (see (†) of 3.1). If  $\mathbf{a} \in A \cap \tilde{U}_x \cap \tilde{U}_y$  then  $U'_x$  and  $\tilde{U}'_y$  are both  $\mathbf{a}$ -small. Since  $U_x$  and  $U_y$  are disjoint we have  $U'_x \cup U'_y = T$  and so  $\mathbf{a} \# \mathbf{a}$  contradicting our Convention (1) of §3.1. Hence  $\tilde{U}_x \cap \tilde{U}_y = \emptyset$  in this case.

If now  $\mathbf{x} \in A$  and  $y \in T$  then by the same reason any  $\mathbf{x}$ -small neighborhood of  $y$  in  $\tilde{T}$  cannot contain  $\mathbf{x}$ . Since every entourage is open in  $\tilde{T}$  we are done in this case too. If finally both points are entourages they coincide with their disjoint neighborhoods. So  $\tilde{T}$  is Hausdorff.

The compactness of  $\tilde{T}$  follows from [Ge1, Proposition D]. □

**Proposition 3.14.** *If a group  $G$  acts 3-discontinuously on a compactum  $T$  then the induced action on  $\tilde{T} = T \sqcup A$  is also 3-discontinuous.*

**Remark.** In [Ge2, Thm 5.1] it is proved that there is a unique topology on the compactified space  $\tilde{T}$  with respect to which the action is 3-discontinuous. The argument below provides a simple proof of this for the induced topology on  $\tilde{T} \subset T \sqcup \text{Ent } T$  introduced above.

*Proof:* For a subset  $X \subset T$  denote by  $\tilde{X} = X \cup \{\mathbf{a} \in A \mid X' \in \text{Small}(\mathbf{a})\} \subset \tilde{T}$  its convex hull in  $\tilde{T}$ . In case if  $X = \{\mathbf{a}\}$  where  $\mathbf{a} \in A$  is an entourage we put  $\tilde{X} = \mathbf{a}$ . For every  $g \in G$  denote by  $\tilde{g}$  its natural extension to  $\tilde{T}$ .

Every point  $x \in \Theta^3 \tilde{T}$  admits a closed neighborhood which is a "cube"  $\tilde{K} = \tilde{X} \times \tilde{Y} \times \tilde{Z}$  where  $X, Y$  and  $Z$  are either disjoint closed subsets of  $T$  or some of  $\tilde{X}, \tilde{Y}, \tilde{Z}$  are isolated entourages (in the latter case we call the corresponding cube *degenerate*). Every compact subset of  $\Theta^3 \tilde{T}$  is a finite union of such cubes. So it is enough to prove that for two cubes  $\tilde{K}_i = \tilde{X}_i \times \tilde{Y}_i \times \tilde{Z}_i \subset \Theta^3 \tilde{T}$  ( $i = 0, 1$ ) the following set is finite:

$$S = \{g \in G \mid : \tilde{g}\tilde{X}_0 \cap \tilde{X}_1 \neq \emptyset, \tilde{g}\tilde{Y}_0 \cap \tilde{Y}_1 \neq \emptyset, \tilde{g}\tilde{Z}_0 \cap \tilde{Z}_1 \neq \emptyset\}.$$

Suppose to the contrary that  $S$  is infinite. Since the action  $G \curvearrowright T$  is 3-discontinuous, every accumulation point of  $S$  with respect to Vietoris topology is a *cross*  $\langle p, q \rangle^\times = p \times T \sqcup T \times q$  [Ge1, Proposition P]. Consider now all possible cases.

*Case 1.* Both cubes are not degenerate, i.e.  $X_i, Y_i, Z_i$  ( $i = 0, 1$ ) are all closed disjoint subsets of  $T$ .

Note that at least one of the "squares"  $X_0 \times X_1$ ,  $Y_0 \times Y_1$  or  $Z_0 \times Z_1$  does not meet the cross. Indeed otherwise two of them intersect both either  $p \times T$  or  $T \times q$  which is impossible as  $X_i, Y_i$  and  $Z_i$  are pairwise disjoint for  $i \in \{0, 1\}$ .

Let us assume that e.g.  $Z_0 \times Z_1 \cap \langle p, q \rangle^\times = \emptyset$ . Let  $g \in S$  be a homeomorphism whose graph is contained in the neighborhood  $T^2 \setminus Z_0 \times Z_1$  of  $\langle p, q \rangle^\times$ . Then  $gZ_0 \cap Z_1 = \emptyset$ . However  $\tilde{g}\tilde{Z}_0 \cap \tilde{Z}_1 \neq \emptyset$ . So there exists  $\mathbf{a} \in \tilde{Z}_0 \setminus Z_0$  such that  $\tilde{g}\mathbf{a} \in \tilde{Z}_1$ . By definition of the convex hull  $Z'_0$  and  $(g^{-1}(Z_1))'$  are  $\mathbf{a}$ -small. Since  $(g^{-1}(Z_1))' \cup Z'_0 = T$  we obtain that  $T$  is the union of two  $\mathbf{a}$ -small sets, so  $\mathbf{a} \# \mathbf{a}$  contradicting our Convention 3.12.

*Case 2.* At least one of the cubes is degenerate.

Then some of the sets  $\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i$  are entourages. Note that since  $\tilde{g}\tilde{X}_0 \cap \tilde{X}_1 \neq \emptyset$  for infinitely many  $g \in S$ , by Dynkin property  $\tilde{X}_0$  and  $\tilde{X}_1$  cannot be entourages simultaneously. The same is true for  $\tilde{Y}_i$  and  $Z_i$  ( $i = 0, 1$ ). So there could be at most 3 entourages among these 6 sets. We consider all the possibilities below.

*Subcase 2.1.* There is only one degenerate cube.

We can assume that  $\tilde{X}_0 = \mathbf{a}$  for some  $\mathbf{a} \in A$ . Then  $\forall g \in S$  we have  $g\mathbf{a} \in \tilde{X}_1$ . So  $g^{-1}X'_1$  is  $\mathbf{a}$ -small. For a limit cross  $\langle p, q \rangle^\times$  for the set  $S$  and  $\mathbf{a}$ -small neighborhoods  $U_p$  and  $U_q$  of the points  $p$  and  $q$  respectively there exists  $g \in S$  such that  $gU'_p \subset U_q$  or  $g^{-1}U'_q \subset U_p$ . If now  $U_q \cap X_1 = \emptyset$  then  $T$  would be the union of  $\mathbf{a}$ -small sets  $g^{-1}X'_1$  and  $g^{-1}U'_q$  contradicting the unlinkness condition  $\mathbf{a} \# \mathbf{a}$ . So for every  $\mathbf{a}$ -small neighborhood  $U_q$  of  $q$  we have  $U_q \cap X_1 \neq \emptyset$ . Since  $X_1$  is closed it follows that  $q \in X_1$ .

At most one of the disjoint sets  $Y_0$  or  $Z_0$  can contain the other point  $p$  of the cross, let  $p \notin Z_0$ . Then for any neighborhood  $U_q$  and for infinitely many elements  $g \in S$  we have  $gZ_0 \subset U_q$ . If  $gZ_0 \cap Z_1 \neq \emptyset$  for infinitely many  $g \in S$  then  $q$  is an accumulation point for  $Z_1$ , and since  $Z_1$  is

closed we obtain that  $q \in Z_1 \cap X_1$  which is impossible. So for almost all  $g \in S : gZ_0 \cap Z_1 = \emptyset$  and this situation has been excluded in Case 1.

*Subcase 2.2.* There are two degenerate cubes.

Note that they cannot belong to the same level, namely if  $\tilde{X}_0 = \mathbf{a} \in A$  and  $\tilde{Y}_0 = \mathbf{b} \in A$  then by the argument of Subcase 2.1 we must have  $q \in Y_1 \cap X_1$  which is impossible.

So let  $\tilde{Y}_1 = \mathbf{b} \in A$  and  $\tilde{X}_0 = \mathbf{a} \in A$ . By the argument of Subcase 2.1 applied now to the inverse elements of  $S$  we obtain that  $p \in Y_0$ . Hence for almost all elements  $g \in S$  we still have  $gZ_0 \cap Z_1 = \emptyset$  which is impossible by Case 1.

*Subcase 2.3.* There are three degenerate cubes.

Then there are at least two of three entourages which are among of the sets of the same level:  $\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i$  ( $i = 0$  or  $i = 1$ ) which is impossible. So neither case can happen. The Proposition is proved.  $\square$

**Lemma 3.15.** *Let  $B$  be an infinite subset of  $A$  and  $C = N_d(B)$  where  $N_d(B)$  is a  $d$ -neighborhood of  $B$  in  $\tilde{T}$ . Then the topological boundaries of  $B$  and  $C$  coincide.*

*In particular, if  $(\mathbf{b}_n)_n$  and  $(\mathbf{c}_n)_n$  are two sequences in  $A$  such that  $d_A(\mathbf{b}_n, \mathbf{c}_n)$  is uniformly bounded, then  $(\mathbf{b}_n)_n$  converges to a point  $p \in T$  if and only if  $\mathbf{c}_n \rightarrow p$ .*

*Proof:* The second claim directly follows from the first one. So to prove the lemma we need only to show that every accumulation point of  $C$  is also an accumulation point of  $B$ . Suppose not and there exists a point  $r \in \partial C \setminus \partial B$ . Then for every neighborhood  $U_r$  of  $r$  in  $\tilde{T}$  there exists an infinite subset  $C_0 \subset C$  such that  $\forall \mathbf{c} \in C_0$  we have  $\mathbf{c} \in U_r$  implying that  $U'_r \subset c$  for some  $c \in \text{Small}(\mathbf{c})$ .

Arguing by induction on  $d$  without loss of generality we may assume that  $d = 1$ . So  $\forall \mathbf{c} \in C \exists \mathbf{b} \in B : \mathbf{c} \# \mathbf{b}$ . Then there exists a subset  $B_0 \subset B$  such that  $d_A(B_0, C_0) \leq 1$ . Since  $C_0$  is infinite by discreteness of  $A$  the set  $B_0$  is infinite too. Let  $p \in T \setminus \{r\}$  be an accumulation point of  $B_0$ . Then for every neighborhood  $U_p$  of  $p$  there exists  $\mathbf{b} \in B_0$ , corresponding to some  $\mathbf{c} \in C_0$ , for which  $U'_p \subset b$  where  $b \in \text{Small}(\mathbf{b})$ . Choosing  $U_p$  to be disjoint from  $U_r$  we obtain  $b \cup c = T$  and so  $\mathbf{b} \bowtie \mathbf{c}$ . A contradiction.  $\square$

**Definition 3.16.** *[Ge1] (Horospheres, Conical and Parabolic Points).* Let  $k$  be a fixed positive integer, and let  $A$  be the above discrete set of entourages.

1) We say that a point  $p \in T$  and an entourage  $\mathbf{e}$  are *neighbors* (with respect to  $A$ ) and write  $\mathbf{e} \#_{A,k} p$ , if there is no  $\mathbf{a} \in A$  such that  $\mathbf{e} - \mathbf{a} - p(k)$ .

2) The *horosphere*  $T_{A,k}(p)$  (or  $T_k(p)$  or  $T(p)$ ) at the point  $p \in T$  is the set

$$T_{A,k}(p) = \{\mathbf{e} \in A \mid \mathbf{e} \#_{A,k} p\}.$$

3) A point  $x \in T$  is called  $(A, k)$ -*conical* (or just *conical*) if  $T_{A,k}(x) = \emptyset$ .

4) A point  $p \in T$  is called  $(A, k)$ -*parabolic* (or just *parabolic*) if  $T_{A,k}(p)$  is infinite.

It is shown in [Ge1] that the notions of  $(A, k)$ -conical and  $(A, k)$ -parabolic points for  $k \geq 3$  (see also Remark 3.5) are equivalent to the standard definitions (see Section 2) of conical and bounded parabolic points respectively.

**Lemma 3.17.** *[Ge1] If the action  $G \curvearrowright T$  is 3-discontinuous and 2-cocompact then every limit point of this action is either conical or bounded parabolic. Furthermore the set of non-conical points is  $G$ -finite and for every parabolic point  $p \in T$  the set  $T(p)$  is  $\text{Stab}_G p$ -finite.*

The next Lemma is proved in [Ge1, Lemma P2] for closed entourages. We prove it below in a general form.

**Lemma 3.18.** *For every  $d > 0$  the parabolic point  $p$  is the unique accumulation point of the  $d$ -neighborhood  $N_d(T_{A,k}(p))$  of the horosphere  $T_{A,k}(p)$ .*

*Proof:* By Lemma 3.15 it is enough to prove the statement for the horosphere  $T_{A,k}(p)$ . Suppose it admits two distinct accumulation points  $p$  and  $q$ . Since the set  $A$  is  $m$ -separating there exists  $\mathbf{a} \in A$  such that  $p - \mathbf{a} - q(k)$  for some  $k \leq m$ . Then by Lemma 3.6 there exists  $\mathbf{b} \in T_{A,k}(p)$  such that  $p - \mathbf{a} - \mathbf{b}(k)$  which is not possible.  $\square$

We have the following transitivity property:

**Lemma 3.19.** *If  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \text{Ent } T$ ,  $p \in T$  and  $k > 2$ . Then  $\mathbf{a} - \mathbf{b} - p(k)$  and  $\mathbf{b} - \mathbf{c} - p(k)$  imply  $\mathbf{a} - \mathbf{c} - p(k)$ .*

*Proof:* If  $a \in \text{Sh}_{\mathbf{a}} \mathbf{b}$  and  $c \in \text{Sh}_{\mathbf{c}} \mathbf{b}$ , then the sets  $b = a', b_1 = c'$  are  $\mathbf{b}$ -small and  $a \cup b = b_1 \cup c = T$ . Then for a  $\mathbf{c}$ -small neighborhood  $c_0$  of  $p$  we have  $\Delta_{\mathbf{c}}(c, c_0) \geq \Delta_{\mathbf{c}}(\text{sh}_{\mathbf{c}} \mathbf{b}, c_0) - \widetilde{\Delta}_{\mathbf{c}}(\text{sh}_{\mathbf{c}} \mathbf{b}, c) > k - 1$ . So  $\Delta_{\mathbf{c}}(c, p) > k - 1 > 0$  and  $p \in b_1$ . Note that  $b \cap b_1 = \emptyset$  since otherwise  $\Delta_{\mathbf{b}}(b_1, \text{sh}_{\mathbf{b}} \mathbf{a}) \leq \widetilde{\Delta}_{\mathbf{b}}(b_1, b \cap b_1) + \Delta_{\mathbf{b}}(b \cap b_1, \text{sh}_{\mathbf{b}} \mathbf{a}) \leq 2$  which is impossible as  $\mathbf{a} - \mathbf{b} - p(k)$  and  $k \geq 2$ . Thus  $b_1 \subset a$  and  $a \cup c = T$ . Since  $c$  was an arbitrary element of  $\text{Sh}_{\mathbf{c}} \mathbf{b}$ , it follows that  $\text{Sh}_{\mathbf{c}} \mathbf{b} \subset \text{Sh}_{\mathbf{c}} \mathbf{a}$  and  $\text{sh}_{\mathbf{c}} \mathbf{a} \subset \text{sh}_{\mathbf{c}} \mathbf{b}$ . Thus  $\Delta_{\mathbf{c}}(\text{sh}_{\mathbf{c}} \mathbf{a}, c_0) > k$ .  $\square$

The above notions allow us to introduce the following relation on the set  $\text{Ent } T$ .

**Definition 3.20.** (**Busemann order**) For  $\mathbf{a}, \mathbf{b} \in \text{Ent } T$ , and  $p \in T$  we say that  $\mathbf{a}$  and  $\mathbf{b}$  are *Busemann ordered* with respect to  $p$  if

$$\text{either } \mathbf{a} = \mathbf{b}, \text{ or } \mathbf{a} - \mathbf{b} - p(k).$$

We will denote this relation by  $\mathbf{a} \geq_{p,k} \mathbf{b}$ .

Lemma 3.19 implies that this relation is a partial order on  $\text{Ent } T$ . Using Busemann order we can reformulate the above definitions of conical and parabolic points as follows.

**Lemma 3.21.** *A point  $p \in T$  is  $A$ -conical if and only if its Busemann order has no minimal elements. A point  $p$  is  $A$ -parabolic if and only if its Busemann order has infinitely many minimal elements.*

### 3.3 Non-refinable tubes.

**Lemma 3.22.** *The set  $\Psi_k(\mathbf{a}, \mathbf{b}) = \{\mathbf{c} \in A : \mathbf{a} - \mathbf{c} - \mathbf{b}(k)\}$  is finite for any  $k \geq 1$ .*

*Proof:* Suppose that  $\mathbf{a} - \mathbf{c} - \mathbf{b}(k)$  and let us prove that  $\mathbf{c} \# (\mathbf{a} \cap \mathbf{b})$ . If it is not true, then we have  $\mathbf{c} \bowtie (\mathbf{a} \cap \mathbf{b})$ , i.e. there exists  $c \in \text{Small}(\mathbf{c})$ ,  $w \in \text{Small}(\mathbf{a} \cap \mathbf{b})$  such that  $c \cup w = T$ . Thus  $c \in \text{Sh}_{\mathbf{c}}\mathbf{a} \cap \text{Sh}_{\mathbf{c}}\mathbf{b}$  and  $\text{sh}_{\mathbf{c}}\mathbf{a} \subset c$ ,  $\text{sh}_{\mathbf{c}}\mathbf{b} \subset c$ . Hence  $\Delta_{\mathbf{c}}(\text{sh}_{\mathbf{c}}\mathbf{a}, \text{sh}_{\mathbf{c}}\mathbf{b}) \leq 1$  which is impossible. It follows that  $\mathbf{c} \# (\mathbf{a} \cap \mathbf{b})$ . The finiteness of  $\Psi_k(\mathbf{a}, \mathbf{b})$  now follows from the discreteness of  $A$   $\square$

**Definition 3.23. (Refinability).** A pair  $\{\mathbf{a}, \mathbf{b}\} \subset A$  is called  $(k-)$ refinable if  $\Psi_k(\mathbf{a}, \mathbf{b}) \neq \emptyset$ , and  $(k-)$ non-refinable otherwise.

The Proposition 3.25 below guarantees the existence of a finite non-refinable tube between two given entourages in  $A$ . To prove it we need the following:

**Lemma 3.24.** *For every integer  $k \geq 2$ , every pair  $\{\mathbf{a}, \mathbf{b}\} \subset A$  is either  $k + 1$ -nonrefinable or there exists  $\mathbf{c} \in \Psi_k(\mathbf{a}, \mathbf{b})$  such that the pair  $\{\mathbf{a}, \mathbf{c}\}$  is  $k + 1$ -nonrefinable.*

*Proof:* Suppose this is not true and let a pair  $\{\mathbf{a}, \mathbf{b}\}$  be a counter-example. By Lemma 3.22 the set  $\Psi_k(\mathbf{a}, \mathbf{b})$  is finite so we can assume in addition that the number  $|\Psi_k(\mathbf{a}, \mathbf{b})|$  is the minimal one among all such counter-examples. So  $\{\mathbf{a}, \mathbf{b}\}$  is  $k + 1$ -refinable and there exists  $\mathbf{c} \in \Psi_{k+1}(\mathbf{a}, \mathbf{b})$  such that the pair  $(\mathbf{a}, \mathbf{c})$  is  $k + 1$ -refinable too. We now claim that

$$\Psi_{k+1}(\mathbf{a}, \mathbf{c}) \subset \Psi_{k+1}(\mathbf{a}, \mathbf{b}) \quad (k > 1). \quad (1)$$

Let  $\mathbf{d} \in \Psi_{k+1}(\mathbf{a}, \mathbf{c})$ . By [Ge1, Lemma T2] we have  $\mathbf{d} - \mathbf{c} - \mathbf{b}(k)$ . Then  $\text{sh}_{\mathbf{d}}\mathbf{b} \subset \text{sh}_{\mathbf{d}}\mathbf{c}$  [Ge2, Lemma B1]. Therefore  $\Delta_{\mathbf{d}}(\text{sh}_{\mathbf{d}}\mathbf{b}, \text{sh}_{\mathbf{d}}\mathbf{a}) \geq \Delta_{\mathbf{d}}(\text{sh}_{\mathbf{d}}\mathbf{c}, \text{sh}_{\mathbf{d}}\mathbf{a})$ . So  $\mathbf{d} \in \Psi_{k+1}(\mathbf{a}, \mathbf{b})$  and (1) follows.

As  $\mathbf{c} \in \Psi_k(\mathbf{a}, \mathbf{b}) \setminus \Psi_k(\mathbf{a}, \mathbf{c})$  we obtain that  $|\Psi_k(\mathbf{a}, \mathbf{c})| < |\Psi_k(\mathbf{a}, \mathbf{b})|$ . Thus by the minimality of  $(\mathbf{a}, \mathbf{b})$  the pair  $(\mathbf{a}, \mathbf{c})$  cannot be a counter-example. Then  $(\mathbf{a}, \mathbf{d})$  is  $(k + 1)$ -nonrefinable. Since  $\mathbf{d} \in \Psi_{k+1}(\mathbf{a}, \mathbf{b}) \subset \Psi_k(\mathbf{a}, \mathbf{b})$  the pair  $(\mathbf{a}, \mathbf{b})$  cannot be a counter-example neither. A contradiction.  $\square$

For a tube  $P = \mathbf{a} - \mathbf{a}_1 - \dots - \mathbf{a}_n - \mathbf{b}$  we denote by  $\partial P$  its boundary  $\{\mathbf{a}, \mathbf{b}\}$ .

**Proposition 3.25.** *For every pair  $\{\mathbf{a}, \mathbf{b}\} \subset A$  and integer  $k \geq 2$  there exists a finite  $k + 2$ -nonrefinable  $k$ -tube  $P \subset A$  such that  $\partial P = \{\mathbf{a}, \mathbf{b}\}$ .*

*Proof:* Suppose this is not true. Let a pair  $\{\mathbf{a}, \mathbf{b}\}$  be a counter-example such that it has the minimal cardinality  $|\Psi_k(\mathbf{a}, \mathbf{b})|$  among all such pairs. Since  $\{\mathbf{a}, \mathbf{b}\}$  is  $k + 2$ -refinable by the above Lemma there exists  $\mathbf{c} \in \Psi_{k+1}(\mathbf{a}, \mathbf{b})$  such that  $\{\mathbf{a}, \mathbf{c}\}$  is  $k + 2$ -nonrefinable. Since the inclusion  $\Psi_k(\mathbf{c}, \mathbf{b}) \subset \Psi_k(\mathbf{a}, \mathbf{b})$  is strict there exists a  $k + 2$ -nonrefinable  $k$ -tube  $Q$  with  $\partial Q = \{\mathbf{c}, \mathbf{b}\}$ . By the transitivity property [Ge1, Lemma T2] the set  $R = \{\mathbf{a}\} \cup Q$  is a  $k$ -tube with the boundary  $\{\mathbf{a}, \mathbf{b}\}$ . It is  $k + 2$ -nonrefinable by construction. Thus the pair  $\{\mathbf{a}, \mathbf{b}\}$  is not a counterexample. We have a contradiction.  $\square$

**Definition 3.26.** [Ge1] (*Horospherical projection*). Let  $p \in \mathcal{P}$  be a parabolic point and  $T(p)$  be a horosphere at  $p$ . Define a projection map  $\Pi_p : A \rightarrow T(p)$  (or  $\Pi_{p,k}$ ) called horospherical projection as follows. If  $\mathbf{a} \notin T_k(p)$  then  $\Pi_p(\mathbf{a}) = \{\mathbf{p} \in T_k(p) : \mathbf{a} - \mathbf{p} = p(k)\}$ ; and if  $\mathbf{a} \in T_k(p)$  then  $\Pi_p(\mathbf{a}) = \mathbf{a}$ .

**Proposition 3.27.** Let  $\mathcal{P}$  denote the set of parabolic points for the action  $G \curvearrowright T$ . Then for any constants  $k > 3$  and  $d > 0$  the following sets are  $G$ -finite:

- 1)  $\forall \{\mathbf{a}, \mathbf{b}\} \subset A : \{\{\mathbf{c}, \mathbf{d}\} \mid \mathbf{c} \in \Pi_p(g\mathbf{a}), \mathbf{d} \in \Pi_p(g\mathbf{b}), p \in \mathcal{P}, g \in G\}$
- 2)  $\mathcal{A}_1 = \{(\mathbf{a}, \mathbf{b}) \mid \Psi_k(\mathbf{a}, \mathbf{b}) = \emptyset, \{\mathbf{a}, \mathbf{b}\} \not\subset T_{A,k}(p), p \in \mathcal{P}\}$ .
- 3) a)  $\{\{p, q\} \subset \mathcal{P} \mid N_d(T_{A,k}(p)) \cap N_d(T_{A,k}(q)) \neq \emptyset\}$ , and  
b)  $\{N_d(T_{A,k}(p)) \cap N_d(T_{A,k}(q)) \mid \{p, q\} \subset \mathcal{P}\}$ .

*Proof:* 1) Suppose to the contrary that the set 1) is infinite. Assume first that  $\mathbf{a} \neq \mathbf{b}$ . Then there exist an infinite sequence of elements  $g_n \in G$ , distinct entourages  $\{\mathbf{c}_n, \mathbf{d}_n\} \subset A$  such that

$$g_n \mathbf{a} - \mathbf{c}_n - p_n(k) \text{ and } g_n \mathbf{b} - \mathbf{d}_n - p_n(k), \mathbf{c}_n \in T_{A,k}(p_n), \mathbf{d}_n \in T_{A,k}(p_n), p_n \in \mathcal{P} \quad (2).$$

Since the set  $\mathcal{P}$  is  $G$ -finite (Lemma 3.17) we can assume that  $p_n = p$ . Since the stabilizer  $\text{Stab}_G p$  acts cofinitely on  $T_{A,k}(p)$  (Lemma 3.17) we can also fix  $\mathbf{c}_n = \mathbf{c} \in T_{A,k}(p)$ , and assume that  $\mathbf{d}_n = h_n(\mathbf{d})$ ,  $\mathbf{d} \in T_{A,k}(p)$ ,  $h_n \in \text{Stab}_G p$ . So (2) gives

$$g_n \mathbf{a} - \mathbf{c} - p(k), g_n \mathbf{b} - \mathbf{d}_n - p(k), \mathbf{c} \in T_{A,k}(p), \mathbf{d}_n \in T_{A,k}(p), p \in \mathcal{P}. \quad (2')$$

The following Lemma implies that  $p$  is a limit point of  $\{g_n \mathbf{b}\}_n$ .

**Lemma 3.28.** If  $\mathbf{b}_n - \mathbf{d}_n - p(k)$  ( $k > 1$ ),  $\mathbf{d}_n \in T_{A,k}(p)$  and  $\lim_{n \rightarrow \infty} \mathbf{d}_n = p$  then  $\lim_{n \rightarrow \infty} \mathbf{b}_n = p$ .

*Proof:* We start with the following.

**Claim.** For every  $k > 1$  there exists  $\mathbf{d} \in T_{A,k}(p)$  such that  $q - \mathbf{d} - p(k)$ .

Indeed by  $m$ -separation property (2) there exists  $\mathbf{a} \in A$  such that  $q - \mathbf{a} - p(k)$  ( $1 < k \leq m$ ). If  $\mathbf{a} \in T_{A,k}$  we are done. If not let  $\mathbf{p} \in \Pi_p(\mathbf{a})$  so  $\mathbf{a} - \mathbf{p} = p(k)$ . Let  $U_p$  be a  $\mathbf{p}$ -small neighborhood of  $p$ . Let also  $b \in \text{Sh}_{\mathbf{p}} \mathbf{a}$ , then  $a \cup b = T$  where  $b$  is  $\mathbf{p}$ -small and  $a = b'$  is  $\mathbf{a}$ -small set respectively. We have  $\Delta_{\mathbf{p}}(b, U_p) \geq \Delta_{\mathbf{p}}(U_p, \text{sh}_{\mathbf{p}} \mathbf{a}) - \tilde{\Delta}(\text{sh}_{\mathbf{p}} \mathbf{a}, b) > k - 1$ . Therefore  $U_p \subset a$  and so  $U_p$  is  $\mathbf{a}$ -small. Then for any  $\mathbf{a}$ -small neighborhood  $U_q$  of  $q$  we have  $\Delta_{\mathbf{a}}(U_q, U_p) > k$ . Hence  $\Delta_{\mathbf{a}}(U_q, a) > k - 1$ . We have proved that  $U_q \subset b$  for any  $b \in \text{Sh}_{\mathbf{p}} \mathbf{a}$ . Thus  $U_q$  is  $\mathbf{p}$ -small and  $U_q \subset \text{sh}_{\mathbf{p}} \mathbf{a}$ . It implies that  $\Delta_{\mathbf{p}}(U_q, U_p) \geq \Delta_{\mathbf{p}}(\text{sh}_{\mathbf{p}} \mathbf{a}, U_p) > k$ . The Claim follows.

*Proof of the Lemma.* Suppose by contradiction that there exists an accumulation point  $q \in T$  of the set  $\{\mathbf{b}_n\}_n$  distinct from  $p$ . By the Claim there exists  $\mathbf{p} \in T_{A,k-1}(p)$  such that  $q - \mathbf{p} - p(k-1)$ . Since  $\mathbf{d}_n \rightarrow p$  and  $\mathbf{b}_n \rightarrow q$  by Lemma 3.6 we obtain  $\mathbf{b}_n - \mathbf{p} - \mathbf{d}_n(k-1)$  ( $n > n_0$ ). So  $\text{sh}_{\mathbf{d}_n} \mathbf{p} \supset \text{sh}_{\mathbf{d}_n} \mathbf{b}_n$ . Since  $\mathbf{b}_n - \mathbf{d}_n - p(k)$  we obtain  $\mathbf{p} - \mathbf{d}_n - p(k-1)$ . This is impossible as  $\mathbf{p} \in T_{A,k-1}(p)$ . The Lemma is proved.  $\square$

It follows from (2') that for any  $(g_n \mathbf{a})$ -small set  $a_n \in \text{Sh}_{\mathbf{c}}(g_n \mathbf{a})$  and a  $\mathbf{c}$ -small neighborhood  $U_p$  of  $p$  we have  $\Delta_{\mathbf{a}_n}(a_n, U_p) > k - 1 > 0$  ( $n \in \mathbb{N}$ ). Thus  $U_p \subset a'_n$  and  $U_p$  is  $g_n \mathbf{a}$ -small for all  $n \in \mathbb{N}$ .

From the other hand by Proposition 3.14 we have that  $G \curvearrowright \tilde{T}$  is a convergence action. Then by [GePo1, Lemma 5.1] for every pair of distinct non-conical points  $\{x, y\} \subset \tilde{T}$  the accumulation points of the orbit  $G(x, y)$  belong to the diagonal  $\Delta^2 \tilde{T}$ . By Lemma 3.28  $\lim_{n \rightarrow \infty} g_n(\mathbf{b}) = p$  so  $\lim_{n \rightarrow \infty} g_n(\mathbf{a}) = p$ . Hence for the above neighborhood  $U_p$  we also have that  $U'_p$  is  $(g_n \mathbf{a})$ -small for some  $n \in \mathbb{N}$ . This is impossible by our Convention (1) of 3.1. Part 1) is proved.  $\square$

2) Suppose that  $\{(\mathbf{a}_i, \mathbf{b}_i) \in A \times A \mid i \in I\}$  is an infinite set such that for every  $i \in I$  there is no  $\mathbf{c}_i \in A$  such that  $\mathbf{a}_i - \mathbf{c}_i - \mathbf{b}_i(k)$ . The set  $A$  is  $G$ -finite so we can fix  $\mathbf{a} = \mathbf{a}_i$  and assume that  $\mathbf{b}_i = g_i(\mathbf{b}) : g_i \in G$ . Since the space  $\tilde{T}$  is compact, the set  $\{\mathbf{b}_i\}_{i \in I}$  admits an accumulation point  $p$  which is a limit point for the geometrically finite action  $G \curvearrowright \tilde{T}$ . By Lemma 3.17  $p$  is either  $k$ -conical or  $k$ -parabolic point for some (any)  $k > 1$ . Consider these two cases separately.

Let first,  $p$  be a  $k$ -conical point. Then there exists  $\mathbf{c} \in A$  such that  $\mathbf{a} - \mathbf{c} - p(k)$ . By Lemma 3.6 we have  $\mathbf{a} - \mathbf{c} - \mathbf{b}_i(k)$  ( $i \in I$ ) contradicting the  $k$ -non-refinability of the pair  $\{\mathbf{a}, \mathbf{b}_i\}$ .

Let us now suppose that  $p$  is  $k$ -parabolic. We will now show that for almost all  $i \in I$  the entourages  $\mathbf{a}$  and  $\mathbf{b}_i$  belong to the same horosphere  $T_{A,k}(p)$ . We claim first that  $\mathbf{a} \in T_{A,k}(p)$ . Indeed if not, then there exists  $\mathbf{c} \in A$  such that  $\mathbf{a} - \mathbf{c} - p(k)$  contradicting by the same argument the  $k$ -non-refinability of the pair  $\{\mathbf{a}, \mathbf{b}_i\}$  ( $i \in I$ ). So  $\mathbf{a} \in T_{A,k}(p)$ .

Suppose by contradiction that there exist  $\mathbf{b}_i \notin T_{A,k}(p)$  for infinitely many  $i \in I$ . Then there exist  $\mathbf{c}_i \in T_{A,k}(p)$  such that

$$\mathbf{b}_i - \mathbf{c}_i - p(k). \quad (*)$$

We first note that in  $(*)$  we cannot have the same entourage  $\mathbf{c}_0$  for infinitely many different  $\mathbf{b}_i$ . Indeed if not, then from  $(*)$  we have  $\Delta_{\mathbf{c}_0}(\text{sh}_{\mathbf{c}_0} \mathbf{b}_i, c_0) > k$  ( $i \in I$ ) for a  $\mathbf{c}_0$ -small set  $c_0$  containing  $p$ . Since  $p$  is an accumulation point for the set  $\{\mathbf{b}_i\}_{i \in I}$  then  $c'_0$  is  $\mathbf{b}_i$ -small for infinitely many  $i \in I$ . Thus  $c_0 \supset \text{sh}_{\mathbf{c}_0} \mathbf{b}_i$ , and  $\Delta_{\mathbf{c}_0}(c_0, \text{sh}_{\mathbf{c}_0} \mathbf{b}_i) \leq 1$  which is impossible.

So we can assume that  $\mathbf{c}_i$  are all distinct. By Lemma 3.17 the quotient  $T_{A,k}(p)/\text{Stab}_G p$  is finite, so there exists  $h_i \in \text{Stab}_G p$  such that  $h_i(\mathbf{c}_i) = \mathbf{c} \in T_{A,k}(p)$ . Hence  $h_i(\mathbf{b}_i) - \mathbf{c} - p(k)$  for every  $i \in I_1$  where  $I_1$  is an infinite subset of  $I$ . Since  $\mathbf{a} \in T_{A,k}(p)$  by Lemma 3.18  $p$  is an accumulation point for the set  $\{h_i(\mathbf{a})\}_{i \in I_1}$ . Then by Lemma 3.6 we obtain  $h_i(\mathbf{b}_i) - \mathbf{c} - h_i(\mathbf{a})(k)$  and so  $\mathbf{b}_i - h_i^{-1} \mathbf{c}_i - \mathbf{a}(k)$  which is impossible.

So  $\mathbf{b}_i \in T_{A,k}(p)$  for almost all  $i \in I$ . This shows that the set  $A_1$  is  $G$ -finite. Part 2) is proved.  $\square$

3) a) We omit the index  $k$  below. Suppose that the first set is infinite. Then there exists an infinite set of  $G$ -non-equivalent pairs of parabolic points  $(p_i, q_i) \in \mathcal{P}^2$  for which  $N_d(T(p_i)) \cap N_d(T(q_i)) \neq \emptyset$  ( $i \in I$ ). Since the action of  $G$  on  $\Theta^2 T$  is cocompact there exist  $g_i \in G$  such that the pair  $(g_i(p_i), g_i(q_i))$  belong to a compact subset of  $\Theta^2 T$ . So without lost of generality we may assume that the sets  $\{p_i\}_{i \in I}$  and  $\{q_i\}_{i \in I}$  admits two distinct accumulation points  $p$  and  $q$ . It follows from [Ge1, Lemma P3] that there cannot exist an entourage belonging to the intersection of infinitely many distinct horospheres (for a more general system of horospheres this is also true, see [GePo3, Corollary of 4.4.2]). So there is an infinite sequence of distinct entourages  $\mathbf{b}_i \in N_d(T(p_i)) \cap N_d(T(q_i))$  ( $i \in I$ ). The set  $\{\mathbf{b}_i\}_{i \in I}$  admits an accumulation point  $x \in T$ . Let



$(\mathbf{c}_i)_i \subset T(p_i)$  and  $(\mathbf{d}_i)_i \subset T(q_i)$  be two subsets for which  $d_A(\mathbf{b}_i, \mathbf{c}_i)$  and  $d_A(\mathbf{b}_i, \mathbf{d}_i)$  are bounded by the constant  $d$ . Thus  $d_A(\mathbf{c}_i, \mathbf{d}_i) \leq 2d$  and by Lemma 3.15 we have  $p = q = x$ . A contradiction.

b) If now the second set is not  $G$ -finite then for a fixed parabolic point  $p \in \mathcal{P}$  by the part a) we obtain  $q \in \mathcal{P}$  such that the set  $N_d(T(p)) \cap N_d(T(q))$  is infinite. Then by 3.15 we must have  $p = q$ . The Proposition is proved.  $\square$

**Corollary 3.29.** *Suppose that  $G$  is a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum  $T$ . Then there exists a constant  $C > 0$  such that the  $d_A$ -diameter of each of the sets 1), 2) and 3b) of Proposition 3.27 is bounded by  $C$ .*

*Proof:* Since  $G$  is finitely generated by Lemma 3.11 the graph  $\mathcal{G}$  is connected. So  $d$  is a real distance. The Corollary follows from the above Proposition.  $\square$

From Proposition 3.27.2 we immediately have:

**Corollary 3.30.** *Let  $G \curvearrowright T$  be a 3-discontinuous and 2-cocompact action satisfying the above conditions. Then if for a fixed  $\mathbf{a} \in A$  and infinitely many  $\mathbf{b}_n \in A$  the pairs  $(\mathbf{a}, \mathbf{b}_n)$  are all non-refinable then for all but finitely many  $n$  one has  $(\mathbf{a}, \mathbf{b}_n) \subset T(p)$ .*  $\square$

We will now obtain few more finiteness properties characterizing the horospherical projection  $\Pi_p : A \rightarrow T_{A,k}(p)$  ( $p \in \mathcal{P}$ ). The following definition is motivated by Lemma 3.6.

**Definition 3.31.** *For a fixed  $k > 3$  a visibility neighborhood of the point  $\mathbf{p} \in \Pi_p(\mathbf{a}) \subset T_{A,k}(p)$  from the point  $\mathbf{a} \in A$  is the following set*

$$\mathcal{N}(\mathbf{a}, \mathbf{p}, p) = \{\mathbf{x} \in T_{A,k}(p) \mid \mathbf{a} - \mathbf{p} - p(k) \wedge \neg \mathbf{a} - \mathbf{p} - \mathbf{x}(k-1)\},$$

where  $\neg$  denotes the opposite logical statement.

The following Proposition establishes the  $G$ -finiteness properties of two more sets (by continuing the notations of 3.27):

**Proposition 3.32.** *For every  $k > 1$  the following sets are  $G$ -finite:*

- 1)  $\mathcal{A}_2 = \{(\mathbf{x}, \mathbf{p}) \in T_k^2(p) \mid \mathbf{x} \in \mathcal{N}(\mathbf{a}, \mathbf{p}, p), \mathbf{a} \in A, p \in \mathcal{P}\}.$
- 2)  $\mathcal{A}_3 = \{\Pi_p(T_k(q)) \mid \{p, q\} \subset \mathcal{P}\}.$

*Proof:* 1) Suppose by contradiction that it is not true and  $\mathcal{A}_2$  is not  $G$ -finite for some  $k > 1$ . Since  $A$  is one  $G$ -orbit up to taking an infinite subset of  $\mathcal{A}_2$  we can fix the entourage  $\mathbf{p}$ . By [Ge1, Lemma P3]  $\mathbf{p}$  can belong to at most finitely many different horospheres. So up to a passing to a new infinite subset we can fix the parabolic point  $p \in \mathcal{P}$ .

If first the set of entourages  $\{\mathbf{a} \mid (\mathbf{x}, \Pi_p(\mathbf{a})) \in \mathcal{A}_2\}$  is finite, up to choosing a new infinite subset of  $\mathcal{A}_2$  we have  $\mathbf{a} - \mathbf{p} - p(k)$  and  $\neg \mathbf{a} - \mathbf{p} - \mathbf{x}(k-1)$  for a fixed  $\mathbf{a}$ . Then the set of the first coordinates  $\{\mathbf{x} \mid (\mathbf{x}, \cdot) \in \mathcal{A}_2\} \subset T(p)$  is infinite and by Lemma 3.18 its accumulation point is  $p$ . Then by Lemma 3.6 there exists  $\mathbf{x}$  in this set such that  $\mathbf{a} - \mathbf{p} - \mathbf{x}(k)$ . A contradiction.

If now the set  $\{\mathbf{a} \mid (\mathbf{x}, \Pi_p(\mathbf{a})) \in \mathcal{A}_2\}$  is infinite let  $q \in T$  be its accumulation point. Taking a  $\mathbf{p}$ -small neighborhood  $U_q$  of  $q$  we obtain that  $U'_q$  is  $\mathbf{a}$ -small for every  $\mathbf{a} \in U_q$ . Thus  $U_q \supset \text{sh}_{\mathbf{p}}\mathbf{a}$ . Since  $\mathbf{a} - \mathbf{p} - p(k)$ , so  $\Delta_{\mathbf{p}}(U_q, U_p) > k - 1$  for a  $\mathbf{p}$ -small neighborhood  $U_p$  of  $p$ . It yields  $q - \mathbf{p} - p(k - 1)$ . There are infinitely many  $\mathbf{x} \in T(p)$  corresponding to the points  $\mathbf{a} \in U_q$ . Since  $p$  is the unique accumulation point of  $T(p)$  we must have  $\mathbf{x} \in U_p$  for most such  $\mathbf{x}$ . Hence  $\Delta(\text{sh}_{\mathbf{p}}\mathbf{a}, \text{sh}_{\mathbf{p}}\mathbf{x}) \geq \Delta_{\mathbf{p}}(U_p, U_q) > k - 1$ . Therefore  $\mathbf{a} - \mathbf{p} - \mathbf{x}(k - 1)$ . Again a contradiction.  $\square$

2) Suppose not. Since the set of parabolic points  $\mathcal{P}$  is  $G$ -finite we can fix the point  $p \in \mathcal{P}$ . Using the action of  $\text{Stab}_G p$  on  $T_k(p)$  we can also assume that there is a fixed entourage  $\mathbf{c} \in T(p)$  such that for every  $q \in \mathcal{P} : \mathbf{c} \in \Pi_p(T(q))$ . So there exists an infinite set  $\{\mathbf{d}_i \in \Pi_p(T(q_i)) \mid i \in I, q_i \in \mathcal{P}\}$  such that for all  $i \in I$  we have

$$\mathbf{b}_i - \mathbf{d}_i - p(k), \mathbf{a}_i - \mathbf{c} - p(k), \{\mathbf{a}_i, \mathbf{b}_i\} \subset T(q_i).$$

Since  $p$  is the unique accumulation point of  $T(p)$ , up to passing to an infinite subsequence of  $I$ , we may assume that  $\lim_{i \rightarrow \infty} \mathbf{d}_i = p$ . Then by Lemma 3.28 we have  $\lim_{i \rightarrow \infty} \mathbf{b}_i = p$ . Let  $q \in T$  be an accumulation point of the set  $\{q_i\}_{i \in I}$ . We claim that  $q = p$ . Indeed if not then there exists an entourage  $\mathbf{a} \in A$  such that  $q - \mathbf{a} - p(k)$ . Hence for infinitely many  $i \in I$  we have  $q - \mathbf{a} - \mathbf{b}_i(k)$  (Lemma 3.6). Then  $\Delta_{\mathbf{a}}(U_q, \text{sh}_{\mathbf{a}}\mathbf{b}_i) > k$  for every  $\mathbf{a}$ -small neighborhood  $U_q$  of  $q$ . So for some  $i \in I$  we have  $q_i \in U_q$  and hence  $q_i - \mathbf{a} - \mathbf{b}_i(k)$ . The latter one is impossible since  $\mathbf{b}_i \in T_k(q_i)$ . If the set  $\{\mathbf{a}_i\}$  has an accumulation point  $r$  different from  $p$  then  $\exists \mathbf{a} \in A : r - \mathbf{a} - p(k)$ . So as above we have  $q_i - \mathbf{a} - \mathbf{a}_i(k)$  which is impossible by the same reason. So  $\lim_{i \rightarrow \infty} \mathbf{a}_i = p$ . Then for every  $\mathbf{c}$ -small neighborhood  $U_p$  of  $p$  we have that  $U_p$  is also  $\mathbf{a}_i$ -small and  $U'_p$  is  $\mathbf{a}_i$ -small for infinitely many  $i$ . This is impossible. The Proposition is proved.  $\square$

The following Corollary gives a uniform bound on the cardinality of the intersection of the stabilizers of parabolic points for a geometrically finite action.

**Corollary 3.33.** *Let  $G$  be a group admitting a 3-discontinuous and 2-cocompact action on a compactum  $T$ . Then there is a constant  $C$  such that for every pair of distinct parabolic points  $p_i$  and  $p_j$  for the action  $G \curvearrowright T$  one has*

$$|\text{Stab}_G p_i \cap \text{Stab}_G p_j| \leq C,$$

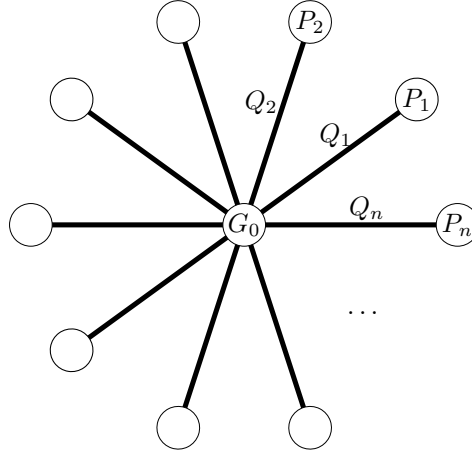
*Proof:* Denote  $H_i = \text{Stab}_G p_i$ . Suppose the statement is not true. By Lemma 3.17 the set of parabolic points for the action  $G \curvearrowright T$  is  $G$ -finite. So up to conjugation we can suppose that there exists a sequence of the stabilizers of parabolic points  $H_0, H_n$  ( $n \in \mathbb{N}$ ) such that  $|H_0 \cap H_n| \rightarrow \infty$ . Let  $T_n$  be a horosphere at  $p_n$  ( $n \in \mathbb{N} \cup \{0\}$ ). Then the projection  $\Pi_{p_0}(T_n)$  of  $T_n$  on  $T_0$  is invariant under  $H_0 \cap H_n$ . Since the action  $H_0 \curvearrowright T_0$  is discontinuous we have  $|\Pi_{p_0}(T_n)| \rightarrow \infty$  which is impossible by Proposition 3.32.2.  $\square$

**Remark.** The above Corollary is also true if  $G$  is a countable group acting 3-discontinuously on a compactum  $T$  such that every point  $T$  is either conical or bounded parabolic. Indeed in this case by [Ge1, Main Theorem.c] the space  $T$  is metrizable. So by [Tu3] the action  $G \curvearrowright T$  is 2-cocompact and the above Corollary holds.

### 3.4 Proof of Theorem A.

The aim of this subsection is the following.

**Theorem A.** *Let  $G$  be a relatively hyperbolic group with respect to a collection of parabolic subgroups  $\{P_1, \dots, P_l\}$ . Then there exists a finitely generated subgroup  $G_0$  of  $G$  which is relatively hyperbolic with respect to the collection  $\{Q_i = P_i \cap G_0 \mid i = 1, \dots, l\}$  such that  $G$  is the fundamental group of the star graph*



whose central vertex group is  $G_0$  and all other vertex groups are  $P_i$  ( $i = 1, \dots, n$ ).

Furthermore for every finite subset  $K \subset G$  the subgroup  $G_0$  can be chosen to contain  $K$ .

*Proof:* Recall that  $A = G(\mathbf{a}_0)$  ( $\mathbf{a}_0 \in \text{Ent } T$ ) is a discrete orbit of entourages forming the vertex set of the graph  $\mathcal{G}$  satisfying our Convention 3.12. Without loss of generality we can assume that the group  $G$  is not finitely generated and  $\mathbf{a}_0 \in K$ . So the graph  $\mathcal{G}$  is not connected (see Lemma 3.11). The distance  $d_A(\mathbf{x}, \mathbf{y})$  is a pseudo-distance being infinity if and only if  $\mathbf{x}$  and  $\mathbf{y}$  belong to different connected components of  $\mathcal{G}$ . By Lemmas 3.17 and 3.18 the set  $\mathcal{P}$  of parabolic points for the action  $G \curvearrowright T$  is  $G$ -finite; and for every  $p \in \mathcal{P}$  the stabilizer  $H_p = \text{Stab}_G p$  acts cofinitely on its horosphere  $T(p)$ .

Let  $\mathcal{A}_i$  ( $i = 1, 2, 3$ )  $\subset A^2$  be the  $G$ -finite sets introduced in Propositions 3.27.2 and 3.32.

We now construct a new graph  $\tilde{\mathcal{G}}$  whose set of vertices is  $A$  and the set of edges is given by the pairs of entourages belonging to the following sets:

- a) the finite set  $K^2$  and the set of all its horospherical projections  $\{\Pi_p(K^2) \mid p \in \mathcal{P}\}$ ;
- b) the set  $\mathcal{A}_1$  and the set of all its horospherical projections  $\{\Pi_p(\mathcal{A}_1) \mid p \in \mathcal{P}\}$ ;
- c) the set  $\mathcal{A}_2$ ;
- d) the set  $\mathcal{A}_3$ .

All these sets are  $G$ -finite. Indeed the set  $\mathcal{A}_1$  is  $G$ -finite by Proposition 3.27.2. So by Proposition 3.27.1 the set  $\{\Pi_p(\mathcal{A}_1) \mid p \in \mathcal{P}\}$  consisting of the projections of finitely many  $G$ -orbits of pairs is  $G$ -finite too. The sets  $\mathcal{A}_2$  and  $\mathcal{A}_3$  are  $G$ -finite by Proposition 3.32.

**Lemma 3.34.** *There exists a finitely generated subgroup  $G_0$  of  $G$  containing any finite subset  $K \subset G$  and which is relatively hyperbolic with respect to  $Q_i = P_i \cap G_0$  ( $i = 1, \dots, n$ ).*

*Proof:* Let  $\mathcal{G}_0$  be the connected component of  $\tilde{\mathcal{G}}$  containing  $K$ . Set  $G_0 = \text{Stab}_G \mathcal{G}_0$  and  $A_0 = \mathcal{G}_0^0$ . By Lemma 3.11 the group  $G_0$  is finitely generated. We are left to prove that  $G_0$  is relatively hyperbolic with respect to the subgroups  $\{Q_i\}_{i=1}^k$ .

Let  $T_0$  be a subset of  $T$  which is the limit set of  $G_0$ . We will first show that the action  $G_0 \curvearrowright T_0$  is 2-cocompact. By [Ge1, Prop. E] the 2-cocompactness is equivalent to the  $k$ -separation property:

$$\forall p, q \in T_0 : p \neq q \exists \mathbf{b} \in A_0 : p - \mathbf{b} - q(k), \quad (1)$$

for some  $k > 0$ . Since the action of  $G$  on  $T$  is 2-cocompact, the property (1) is true for some  $\mathbf{b} \in A$ . If  $\mathbf{b} \in A_0$  we are done, so suppose that  $\mathbf{b} \notin A_0$ . Let  $U_p$  and  $U_q$  be  $\mathbf{b}$ -small neighborhoods of the points  $p$  and  $q$  such that  $\Delta_{\mathbf{b}}(U_p, U_q) > k$ . Since  $p$  and  $q$  are accumulation points of  $A_0$  there exist entourages  $\mathbf{a}, \mathbf{c} \in A_0$  such that  $U'_p$  is  $\mathbf{a}$ -small and  $U'_q$  is  $\mathbf{c}$ -small. So  $U_p \supset \text{sh}_{\mathbf{b}} \mathbf{a}$  and  $U_q \supset \text{sh}_{\mathbf{b}} \mathbf{c}$ . Hence

$$\mathbf{a} - \mathbf{b} - \mathbf{c}(k). \quad (2)$$

By Proposition 3.25 up to refining the pair  $\{\mathbf{a}, \mathbf{b}\}$  we can suppose that the pair  $\{\mathbf{a}, \mathbf{b}\}$  is  $k+2$ -nonrefinable. Since  $\mathbf{b} \notin A_0$ , by operation b) above the pair  $\{\mathbf{a}, \mathbf{b}\}$  must belong to an horosphere  $T_{k+2}(r)$  ( $r \in \mathcal{P}$ ). As  $\{\mathbf{a}, \mathbf{c}\} \subset A_0$  and  $\mathcal{G}_0$  is connected there exists a path  $\gamma = \gamma(\mathbf{a}, \mathbf{c}) \subset \mathcal{G}_0$ . Let  $\mathbf{e} = \Pi_r(\mathbf{c})$ . Note that for every edge  $l \in \mathcal{G}_0^1$  we have  $\Pi_r(l) \in \mathcal{G}_0^1$ . Indeed if  $l$  joins two vertices of  $A_0$  then by the operations a), b) and d) all their horospherical projections are joined by edges too. So  $\Pi_r(\mathcal{G}_0) \subset \mathcal{G}_0$ . Since  $\{\mathbf{a}, \mathbf{e}\} \subset T(r) \cap \Pi_r(\gamma)$  we have  $\mathbf{e} \in A_0$ .

Operation c) then implies that  $\mathbf{b} \notin \mathcal{N}(\mathbf{c}, \mathbf{e}, r)$ . By Definition 3.31 we have

$$\mathbf{b} - \mathbf{e} - \mathbf{c}(k+1). \quad (3)$$

So  $\text{sh}_{\mathbf{b}} \mathbf{c} \subset \text{sh}_{\mathbf{b}} \mathbf{e}$  and (2) yields  $\Delta_{\mathbf{b}}(\text{sh}_{\mathbf{b}} \mathbf{a}, \text{sh}_{\mathbf{b}} \mathbf{e}) > k-1$  and  $\mathbf{a} - \mathbf{b} - \mathbf{e}(k-1)$ . Thus  $\text{sh}_{\mathbf{e}} \mathbf{a} \subset \text{sh}_{\mathbf{e}} \mathbf{b}$  and by (3) we have  $\mathbf{a} - \mathbf{e} - \mathbf{c}(k-1)$  with  $\mathbf{e} \in A_0$ . We have proved that the action  $G_0 \curvearrowright T_0$  is  $(k-1)$ -separating and so is 2-cocompact [Ge1, Prop. E].

By [Ge1, Main Thm] every point of  $T_0$  is either conical or parabolic for the action of  $G_0$  on  $T_0$ . Let  $p \in T_0$  be a parabolic point for this action. We need the following.

**Claim.** *The point  $p$  is also parabolic for the action of  $G$  on  $T$ .*

*Proof of the Claim.* Suppose not. Let  $T(p) \subset A_0$  be a horosphere for the action  $G_0 \curvearrowright \mathcal{G}_0$ . Let us choose  $\mathbf{b} \in T_{k-3}(p) \subset T(p)$  ( $k > 3$ ) where  $T_s(\cdot)$  denotes the "sub-horosphere" of  $T(\cdot)$  of order  $s$  (see 3.16).

Suppose first that  $\mathbf{b}$  does not belong to any horosphere  $\tilde{T}_k(q) \subset A$  for the action  $G \curvearrowright \tilde{\mathcal{G}}$ . Since  $p$  is conical for the action  $G \curvearrowright T$  there exists  $\mathbf{c} \in A$  such that  $\mathbf{b} - \mathbf{c} - p(k-1)$ . Note that  $\mathbf{c} \notin A_0$  as otherwise  $\mathbf{b} \notin T_{k-1}(p)$  which is impossible as  $T_{k-3}(p) \subset T_{k-1}(p)$ . By the Sublemma below we can also suppose up to refining the couple  $(\mathbf{b}, \mathbf{c})$  that it is not  $k$ -refinable ( $k > 3$ ). Since  $\mathbf{b}$  and  $\mathbf{c}$  do not belong to one horosphere in  $\tilde{\mathcal{G}}$ , by operation b) above,  $\mathbf{c}$  and  $\mathbf{b}$  are joined by an edge in  $\tilde{\mathcal{G}}$ . So  $\mathbf{c} \in A_0$  and we have a contradiction in this case.

We affirm now that there exists  $h \in \text{Stab}_{G_0}p$  such that  $h(\mathbf{b})$  does not belong to any horosphere  $\tilde{T}_k(q)$  where  $q \in \mathcal{P}$ . Suppose not, then  $\mathbf{b} \in T_{k-3}(p) \cap \tilde{T}_k(q)$  for some  $q \in \mathcal{P}$ . Again since  $p$  is conical for the action on  $T$  there exists  $\mathbf{c} \in A \setminus A_0$  such that  $\mathbf{b} - \mathbf{c} - p(k-1)$ . By the argument above we can assume that  $\mathbf{c} \in \tilde{T}_k(q)$  too. Up to choosing  $h \in \text{Stab}_{G_0}p$  so that  $\mathbf{b}_1 = h(\mathbf{b}) \in T_{k-3}(p)$  is sufficiently close to  $p$  we can also assume that  $\mathbf{b} - \mathbf{c} - \mathbf{b}_1(k-1)$  (Lemma 3.6). As the distance  $d_{A_0}(\mathbf{b}, \mathbf{b}_1)$  is large, by Proposition 3.27.3b we have that  $\mathbf{b}_1 \notin \tilde{T}_k(q)$ . Then there exists  $q_1 \in \mathcal{P} \setminus \{p, q\}$  such that  $\mathbf{b}_1 \in \tilde{T}_k(q_1)$ . By the argument above giving the formula (3) it follows that there exists  $\mathbf{e} \in \Pi_q(\mathbf{b}_1) \cap A_0$  such that  $\mathbf{b}_1 - \mathbf{e} - \mathbf{c}(k-1)$  and so  $\mathbf{b} - \mathbf{e} - \mathbf{b}_1(k-2)$ . Continuing in this way we obtain an infinite sequence  $\mathbf{b}_n = h_n(\mathbf{b}) \in T_{k-3}(p) \cap \tilde{T}_k(q_n)$  where  $h_n \in \text{Stab}_{G_0}p$  and  $q_n = h_n(q)$  are all different parabolic points. By Proposition 3.27.1 it follows that the subset  $B = \bigcup_{n \in \mathbb{N}} \Pi_q(h_n(\mathbf{b}))$  of  $\tilde{T}_k(q) \cap A_0$  is finite. So up to choosing a new subsequence for a fixed  $\mathbf{e} \in B$  we have  $\mathbf{b} - \mathbf{e} - \mathbf{b}_n(k-2)$  ( $n \in \mathbb{N}$ ). Since  $p$  is the accumulation point of  $\{\mathbf{b}_n\}_{n \in \mathbb{N}}$ , for any  $\mathbf{e}$ -small neighborhood  $U_p$  of  $p$  its complement  $U'_p$  is  $\mathbf{b}_n$ -small for infinitely many  $n$ . Thus  $U_p \supset \text{sh}_{\mathbf{e}}\mathbf{b}_n$  and so  $\Delta_{\mathbf{e}}(\text{sh}_{\mathbf{e}}\mathbf{b}, U_p) > k-3$  implying  $\mathbf{b} - \mathbf{e} - p(k-3)$ . Therefore  $\mathbf{b} \notin T_{k-3}(p)$  which is a contradiction proving the Claim.  $\square$

We have  $\text{Stab}_{G_0}p = \text{Stab}_Gp \cap G_0$ . Lemma 3.34 is proved modulo the following Lemma.

**Sublemma 3.35.** *If  $\mathbf{b} - \mathbf{c} - p(k-1)$  and  $\mathbf{b} - \mathbf{c}_1 - \mathbf{c}(k)$  then  $\mathbf{b} - \mathbf{c}_1 - p(k-1)$  ( $k > 3$ ).*

*Proof:* Let us first show that  $\mathbf{c}_1 - \mathbf{c} - p(k-2)$ . Indeed the second assumption implies that  $\text{sh}_{\mathbf{c}}\mathbf{c}_1 \supset \text{sh}_{\mathbf{c}}\mathbf{b}$ . So for a  $\mathbf{c}$ -small neighborhood  $U_p$  of  $p$  using the first assumption for any  $c \in \text{Sh}_{\mathbf{c}}\mathbf{c}_1$  we have

$$\Delta_{\mathbf{c}}(c, U_p) > \Delta_{\mathbf{c}}(\text{sh}_{\mathbf{c}}\mathbf{b}, U_p) - \tilde{\Delta}_{\mathbf{c}}(\text{sh}_{\mathbf{c}}\mathbf{b}, c) > k-2.$$

So  $U_p \subset c' \in \text{Sh}_{\mathbf{c}_1}\mathbf{c}$  and  $\tilde{\Delta}_{\mathbf{c}_1}(\text{sh}_{\mathbf{c}_1}\mathbf{c}, U_p) \leq 1$ . Hence  $\Delta_{\mathbf{c}_1}(\text{sh}_{\mathbf{c}_1}\mathbf{b}, U_p) > \Delta_{\mathbf{c}_1}(\text{sh}_{\mathbf{c}_1}\mathbf{b}, \text{sh}_{\mathbf{c}_1}\mathbf{c}) - \Delta_{\mathbf{c}_1}(\text{sh}_{\mathbf{c}_1}\mathbf{c}, U_p) > k-1$ . The Lemma and the Proposition are proved.  $\square$

The following Lemma finishes the proof of the Theorem.

**Lemma 3.36.** *The action  $G \curvearrowright \tilde{\mathcal{G}}$  induces an action on a bipartite simplicial tree  $\mathcal{T}$  such that the graph  $X = \mathcal{T}/G$  satisfies Theorem A.*

*Proof:* Using the graph  $\tilde{\mathcal{G}}$  we construct the tree  $\mathcal{T}$  to have vertices belonging to two subsets  $\mathcal{C}$  and  $\mathcal{H}$ . The elements of  $\mathcal{C}$  are components of  $\tilde{\mathcal{G}}$  and the elements of  $\mathcal{H}$  are the horospheres of  $A = \tilde{\mathcal{G}}^0$ . We call them *non-horospherical* and *horospherical* respectively. Two vertices  $C$  and  $H$  of  $\mathcal{T}$  are joined by an edge if and only if  $C \in \mathcal{C}$ ,  $H \in \mathcal{H}$ , and  $C \cap H \neq \emptyset$ .

Let us first show that  $\mathcal{T}$  is connected. Indeed by construction every horospherical vertex is joined with a non-horospherical one. So it is enough to prove that every two non-horospherical vertices can be joined by a path. Let  $C_i$  ( $i = 1, 2$ ) be the corresponding connected components of  $\tilde{\mathcal{G}}$  and let us fix two entourages  $\mathbf{a} \in C_1^0$  and  $\mathbf{b} \in C_2^0$ . By Proposition 3.25 there exists a non-refinable tube between them:  $P = \mathbf{a} - \mathbf{b}_1 - \dots - \mathbf{b}_n - \mathbf{b} \subset A$ . By operation b) above every non-refinable pair  $(\mathbf{b}_i, \mathbf{b}_{i+1})$  either belongs to an horosphere  $T(p)$  or corresponds to an edge in the graph  $\tilde{\mathcal{G}}$ . In the latter case it stays in the same component of  $\tilde{\mathcal{G}}$ . In the former case the

horosphere  $T(p)$  corresponds to a single vertex of the graph  $\mathcal{T}$ . So the tube  $P$  produces a path in  $\mathcal{T}$  between the corresponding vertices. Thus  $\mathcal{T}$  is connected.

Let us now show that  $\mathcal{T}$  is a tree. Suppose not and it contains a simple loop  $\alpha$ . Since the vertices of two types alternate on  $\alpha$  we can fix a horospherical vertex  $H$  corresponding to the horosphere  $T(p)$  and having two non-horospherical neighboring vertices  $C_1$  and  $C_2$ . Let  $\alpha_1$  be a subpath of  $\alpha$  containing the vertices  $H, C_1, C_2$ , and  $\alpha_2$  be the closure of  $\alpha \setminus \alpha_1$ . The path  $\alpha_2$  corresponds to an alternating sequence of components of  $\tilde{\mathcal{G}}$  and horospheres. So we can choose a sequence of tubes  $P_i \subset C_i$  where each  $C_i$  ( $i \geq 3$ ) is a component of  $\tilde{\mathcal{G}}$  corresponding to a non-horospherical vertex of  $\alpha_2$ . The tube  $P_i$  connects two entourages from  $C_i$  each belonging to horospheres  $T(q_i)$  and  $T(q'_i)$  intersecting  $C_i$ . Note that these horospheres differ from the initial horosphere  $T(p)$  as  $\alpha$  is a simple loop. By operations b) and d) above it follows that there exists the path  $\bigcup_i \Pi_p(P_i \cup T(q_i) \cup T(q'_i))$  on  $T(p) \cap \tilde{\mathcal{G}}$ . It implies that the vertices  $C_1$  and  $C_2$

correspond to the same connected component of  $\tilde{\mathcal{G}}$  which is impossible. So  $\mathcal{T}$  is a tree.

By Lemma 3.34 we can assume that the stabilizer  $G_0$  of a component  $\mathcal{G}_0 \in \mathcal{C}$  is finitely generated and contains the fixed finite set  $K \subset G$ . The group  $G$  acts transitively on  $A$  and so on  $\mathcal{C}$ . Then every element of  $\mathcal{C}$  is stabilized by a subgroup conjugate to  $G_0$ . So in the graph  $X = \mathcal{T}/G$  there is only one non-horospherical vertex  $v_0 = \mathcal{C}/G$  whose vertex group is  $G_0$ .

The set of horospheres on  $T$  is  $G$ -finite (Lemma 3.17) so  $X$  contains  $n$  vertices of non horospherical type each representing the  $G$ -orbit of an horosphere  $T(p)$  ( $p \in \mathcal{P}$ ). Every one of them is connected with  $v_0$  by a unique edge. So every vertex group of horospherical type is  $P_i$  and the edge groups are  $Q_i = P_i \cap G_0$  ( $i = 1, \dots, n$ ). The Theorem is proved.  $\square$

Theorem A admits several immediate corollaries describing different type of finiteness properties of relatively hyperbolic groups.

**Corollary 3.37.** *Let  $G$  be a relatively hyperbolic group with respect to the system  $P_j$  ( $j = 1, \dots, n$ ). Then there exists an exhaustion  $G = \bigcup_{i \in I} G_i$  where  $G_i$  is a finitely generated relatively hyperbolic group with respect to  $P_j \cap G_i$  ( $j = 1, \dots, n$ ).*  $\square$

**Definition 3.38.** *A group  $G$  is called finitely generated with respect to subgroups  $H_i$  ( $i \in I$ ) if it is generated by a finite set  $S$  and the subgroups  $H_i$ .*

**Corollary 3.39.** *Let  $G$  be a group acting 3-discontinuously and 2-cocompactly on a compactum  $T$ . Then  $G$  is finitely generated with respect to the stabilizers of the parabolic points.*  $\square$

**Corollary 3.40.** *A group  $G$  acting 3-discontinuously and 2-cocompactly on a compactum  $T$  without parabolic points is finitely generated.*  $\square$

**Remark.** If in particular  $G$  acts 3-discontinuously and 3-cocompactly on  $T$  without isolated points then every point of  $T$  is conical [GePo1, Appendix]. So by Corollary 3.40  $G$  is finitely generated in this case. By a direct argument one can now deduce that  $G$  is word-hyperbolic [GePo1, Appendix]. This provides a new proof of a theorem due to B. Bowditch [Bo3].  $\square$

## 4 Floyd metrics and shortcut metrics.

From now on we will assume that  $G$  is a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum  $T$ . Let us first recall few standard definitions concerning Floyd compactification (see [F], [Ka], [Tu1], [Ge2], [GePo1] for more details).

We will deal with abstract graphs even without assuming any group action (in particular it can be the Cayley graph or the entourage graph  $\mathcal{G}$  considered in Section 3).

Let  $\Gamma$  be a locally finite connected graph. For a finite path  $\alpha : I \rightarrow \Gamma$  ( $I \subset \mathbb{Z}$ ) we define its *length* to be  $|I| - 1$ . We denote by  $d(\cdot, \cdot)$  the canonical shortest path distance function on  $\Gamma$ , and by  $B(v, R)$  the ball at a vertex  $v \in \Gamma^0$  of radius  $R$ .

Let  $f : \mathbb{N} \rightarrow \mathbb{R}_{>0}$  be a function satisfying the following conditions :

$$\exists \lambda > 0 \forall n \in \mathbb{N} : 1 < \frac{f(n)}{f(n+1)} < \lambda \quad (1)$$

$$\sum_{n \in \mathbb{N}} f(n) < +\infty. \quad (2)$$

Define the Floyd length  $L_{f,v}(\alpha)$  of a path  $\alpha = \alpha(a, b) \subset \Gamma$  with respect to a vertex  $v$  as follows:

$$L_{f,v}(\alpha) = \sum_i f(d(v, \{x_i, x_{i+1}\})). \quad (*)$$

where  $\alpha^0 = \{x_i\}_i$  is the set of vertices of  $\alpha$  (we assume  $f(0) := f(1)$  to make it well-defined).

The Floyd metric  $\delta_{f,v}$  is defined to be the corresponding shortest path metric:

$$\delta_{f,v}(a, b) = \inf_{\alpha} L_{f,v}(\alpha), \quad (**)$$

where the infimum is taken over all paths  $\alpha$  between the vertices  $a$  and  $b$  in  $\Gamma$ . We denote by  $\overline{\Gamma}_f$  the Cauchy completion of the metric space  $(\Gamma, \delta_{f,v})$  and call it *Floyd completion*. Let

$$\partial_f \Gamma = \overline{\Gamma}_f \setminus \Gamma$$

be its boundary, called *Floyd boundary*.

If  $\Gamma$  is a Cayley graph  $\mathcal{Ca}(G, S)$  of a group  $G$  with respect to a finite generating system  $S$  we denote by  $\overline{G}_f$  and by  $\partial_f G$  the Floyd completion and the Floyd boundary respectively. Then the condition (1) above implies that the  $G$ -action extends to its Floyd completion  $\overline{G}_f$  by homeomorphisms [Ka]. Therefore in this case for any  $g \in G$  the Floyd metric  $\delta_g$  is the  $g$ -shift of  $\delta_1$ :

$$\delta_g(x, y) = \delta_1(g^{-1}x, g^{-1}y), \quad x, y \in \overline{G}_f, g \in G,$$

where 1 is the neutral element of  $G$ . Every two metrics  $\delta_{g_1}$  and  $\delta_{g_2}$  are bilipshitz equivalent with a Lipshitz constant depending on  $d(g_1, g_2)$ . The same properties are valid for any locally finite, connected and  $G$ -finite graph  $\Gamma$  ( $|\Gamma^0/G| < \infty$ ).

Recall that a *quasi-isometric map* (or *c-quasi-isometric map*)  $\varphi : X \rightarrow Y$  between two metric spaces  $X$  and  $Y$  is a correspondence such that :

$$\frac{1}{c}d_X(x, y) - c < d_Y(\varphi(x), \varphi(y)) \leq cd_X(x, y) + c,$$

where  $c$  is a uniform constant and  $d_X, d_Y$  denote the metrics of  $X$  and  $Y$  respectively.

If in addition  $d_X(\text{id}, \psi \circ \varphi) \leq \text{const}$  for a  $(c)$ -quasi-isometric map  $\psi : Y \rightarrow X$  we say that  $\varphi$  is a  $(c)$ -quasi-isometry between  $X$  and  $Y$ .

A  $c$ -quasi-isometric map  $\varphi : I \rightarrow X$  is called *c-quasigeodesic* if  $I$  is a convex subset of  $\mathbb{Z}$  or  $\mathbb{R}$ . A quasigeodesic path  $\gamma : I \rightarrow \Gamma$  defined on a half-infinite subset  $I$  of  $\mathbb{Z}$  is called *(quasi-)geodesic ray*; a (quasi-)geodesic path defined on the whole  $\mathbb{Z}$  is called *(quasi-)geodesic line*.

The following Lemma will be often used.

**Lemma 4.1.** (Karlsson Lemma). *Let  $\Gamma$  be a locally finite connected graph. Then for every  $\varepsilon > 0$  and every  $c > 0$ , there exists a finite set  $D$  such that  $\delta_v$ -length of every  $c$ -quasigeodesic  $\gamma \subset \Gamma$  that does not meet  $D$  is less than  $\varepsilon$ .*  $\square$

**Remark.** A. Karlsson [Ka] proved it for geodesics in the Cayley graphs of finitely generated groups. The proof of [Ka] does not use the group action and is also valid for quasigeodesics.

Consider now a set  $S$  of paths of the form  $\gamma : [0, n] \rightarrow \Gamma$  of unbounded length starting at the point  $a = \alpha(0) \in \Gamma$ . Every  $\gamma \in S$  can be considered as an element of the product  $\prod_{i \in I} B(a, i)$ . Since  $\Gamma$  is a locally finite graph the latter space is compact in the Tikhonov topology. So every infinite sequence  $(\alpha_n)_n \subset S$  possesses a “limit path”  $\delta : [0, +\infty) \rightarrow \Gamma$  whose initial segments are initial segments of  $\alpha_n$ .

The following Lemma illustrates the properties of limits of infinite quasigeodesics of  $\Gamma$ .

**Lemma 4.2.** [GePo1] *Let  $\Gamma$  be a locally finite connected graph. Then the following statements are true:*

- 1) *Every infinite ray  $r : [0, +\infty[ \rightarrow \Gamma$  converges to a point at the boundary:  $\lim_{n \rightarrow \infty} r(n) = p \in \partial_f \Gamma$ .*
- 2) *For every point  $p \in \partial_f \Gamma$  and every  $a \in \Gamma$  there exists a geodesic ray joining  $a$  and  $p$ .*
- 3) *Every two distinct points in  $\partial_f \Gamma$  can be joined by a geodesic line.*

$\square$

Let  $\Gamma$  be a locally finite, connected graph on which a finitely generated group  $G$  acts co-compactly. Besides the Floyd metrics the Floyd completion  $\overline{\Gamma}_f$  possesses a set of *shortcut* pseudometrics which can be introduced as follows (see also [Ge2], [GePo1]). Let  $\omega$  be a closed  $G$ -invariant equivalence relation on  $\overline{\Gamma}_f$ . Then there is an induced  $G$ -action on the quotient space  $\overline{\Gamma}_f / \omega$ . A shortcut pseudometric  $\overline{\delta}_g$  is the maximal element in the set of symmetric functions  $\varrho : \overline{\Gamma}_f \times \overline{\Gamma}_f \rightarrow \mathbb{R}_{\geq 0}$  that vanish on  $\omega$  and satisfy the triangle inequality, and the inequality  $\varrho \leq \delta_g$ .



For  $p, q \in \bar{\Gamma}_f$  the value  $\bar{\delta}_g(p, q)$  is the infimum of the finite sums  $\sum_{i=1}^n \delta_g(p_i, q_i)$  such that  $p = p_1$ ,  $q = q_n$  and  $\langle q_i, p_{i+1} \rangle \in \omega$  ( $i=1, \dots, n-1$ ) [BBI, pp 77]. Obviously, the shortcut pseudometric  $\bar{\delta}_g$  is the  $g$ -shift of  $\bar{\delta}_1$ . The metrics  $\bar{\delta}_{g_1}$ ,  $\bar{\delta}_{g_2}$  are bilipschitz equivalent for the same constant as for  $\delta_{g_1}$ ,  $\delta_{g_2}$ .

The pseudometric  $\bar{\delta}_g$  is constant on  $\omega$ -equivalent pairs of points of  $\partial_f \Gamma$ , so it induces a pseudometric on the quotient space  $\bar{\Gamma}_f / \omega$ . We denote this induced pseudometric by the same symbol  $\bar{\delta}_g$ .

Let  $\Gamma$  be a connected, locally finite and  $G$ -finite graph. The graph  $\mathcal{G}$  given by the discrete system  $A = G(\mathbf{a}_0)$  ( $\mathbf{a}_0 \in \text{Ent } T$ ) of entourages (see Definition A and Convention 3.12) is also locally finite,  $G$ -finite and connected (Lemma 3.11). So there exists a  $c$ -quasi-isometry  $\varphi : \Gamma \rightarrow \mathcal{G}$ . Let  $f$  and  $g$  be scaling functions satisfying (1-2) and the condition:

$$\frac{g(n)}{f(cn)} < D \quad (n \in \mathbb{N}), \quad (3)$$

where  $c$  is the above quasi-isometry constant. By [GePo1, Lemma 2.5] the map  $\varphi$  extends to a  $G$ -equivariant Lipschitz map between the Floyd completions  $\bar{\Gamma}_f$  and  $\bar{\mathcal{G}}_g$  of these graphs. We denote this map by the same letter  $\varphi$ . The following Lemma is a direct consequence of the main result of [Ge2]:

**Lemma 4.3.** (Floyd map). Let  $G$  be a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum  $T$ . Then there exist  $\mu \in ]0, 1[$  and a continuous  $G$ -equivariant map  $F : \bar{\Gamma}_f \rightarrow \tilde{T} = A \sqcup T$  for the scaling function  $f(n) = \mu^n$ .

Furthermore for every vertex  $v \in \Gamma^0$  the quantity  $\bar{\delta}_{\mathbf{v}}(F(x), F(y))$  is a metric on  $\tilde{T}$  where  $x, y \in \bar{\Gamma}_f$  and  $\mathbf{v} = \varphi(v) = F(v)$ .

*Proof:* It follows from [Ge2] that there exists  $\nu \in ]0, 1[$  and a continuous  $G$ -equivariant map  $\mathcal{F} : \bar{\mathcal{G}}_g \rightarrow \tilde{T}$  where  $g(n) = \nu^n$ .

Let  $\varphi : \bar{\Gamma}_f \rightarrow \bar{\mathcal{G}}_g$  be the  $G$ -equivariant Lipschitz map described above where  $f(n) = \mu^n$  and  $\mu = \nu^{1/c}$ . Set  $F = \mathcal{F} \circ \varphi$ . The map  $F$  transfers the pseudometric  $\bar{\delta}_v$  on  $\bar{\Gamma}_f$  to  $\tilde{T}$  as follows:

$$\bar{\delta}_{\mathbf{v}}(F(x), F(y)) = \bar{\delta}_v(x, y), \text{ where } \mathbf{v} = F(v), v \in \mathcal{Ca}(G, S).$$

By [Ge2] each  $\bar{\delta}_{\mathbf{v}}$  is a metric on  $\tilde{T}$ . The kernel of  $F$  is the closed  $G$ -invariant equivalence relation on  $\bar{\Gamma}_f$  such that  $\bar{\delta}_{\mathbf{v}}(F(x), F(y)) = 0$ . Indeed since  $\bar{\delta}_{\mathbf{v}}$  is a metric on  $\tilde{T}$  the latter one yields  $F(x) = F(y)$  ( $x, y \in \bar{\Gamma}_f$ ).

□

**Remarks 4.4.** 1) We will call the obtained metric  $\bar{\delta}_{\mathbf{v}}$  ( $\mathbf{v} = F(v) \in A$ ) on  $\tilde{T}$  shortcut (Floyd) metric.

2) Lemma 4.3 is in particular true for any polynomial scalar function  $f$ . Moreover one can put  $f = g$  as  $f(cn)/f(n) = \text{const}$  in this case.

3) Since  $\bar{\delta}_g \leq \delta_g$  the Karlsson Lemma 4.1 is also true when one replaces the Floyd  $\delta_v$ -length by the shortcut  $\bar{\delta}_g$ -length.

## 5 Horospheres and tubes.

Let a finitely generated group  $G$  act 3-discontinuously and 2-cocompactly on a compactum  $T$ . Then the graph of entourages  $\mathcal{G}$  is connected (Lemma 3.11). We will use the graph distance  $d_A$  on  $\mathcal{G}$  as well as the set of shortcut metrics  $\bar{\delta}_{\mathbf{v}}$  ( $\mathbf{v} \in \mathcal{G}$ ) on the compactified space  $\tilde{T} = T \cup A$  coming from Lemma 4.3 where  $A = \mathcal{G}^0$ .

We obtain in this Section several properties of tubes and horospheres which will be used later on.

**Lemma 5.1.** *For any integer  $k > 1$  there exists a constant  $\nu > 0$  such that*

$$\forall \mathbf{a}, \mathbf{c} \in \tilde{T} = T \sqcup A, \forall \mathbf{b} \in A : \mathbf{a} - \mathbf{b} - \mathbf{c}(k) \text{ then } \bar{\delta}_{\mathbf{b}}(\mathbf{a}, \mathbf{c}) \geq \nu.$$

*Proof:* For a fixed entourage  $\mathbf{b} \in A$  let  $C_{\mathbf{b},k}$  denote the closure of the set  $\{\{\mathbf{a}, \mathbf{c}\} \in \tilde{T} \times \tilde{T} : \mathbf{a} - \mathbf{b} - \mathbf{c}(k)\}$  in  $\tilde{T}$ . We first claim that the set  $C_{\mathbf{b},k}$  does not intersect the diagonal of  $\tilde{T} \times \tilde{T}$ . Suppose not and  $(p, p) \in C_{\mathbf{b},k} \cap \Delta^2 \tilde{T}$ . Then there exist two infinite sequences  $(\mathbf{a}_n)_n$  and  $(\mathbf{c}_n)_n$  in  $C_{\mathbf{b},k}$  converging to  $p$ . By discreteness of  $A$  we may suppose that  $p \in T$ . By Lemma 3.6 we have  $\mathbf{a}_n - \mathbf{b} - \mathbf{c}_n(k)$ . Let  $U$  be a  $\mathbf{b}$ -small neighborhood of  $p$ . Then  $U'$  is  $\mathbf{a}_n$ -small and  $\mathbf{c}_n$ -small simultaneously for  $n > n_0$ . Hence  $\text{sh}_{\mathbf{b}} \mathbf{a}_n \cup \text{sh}_{\mathbf{b}} \mathbf{c}_n \subset U$ , and so  $\Delta_{\mathbf{b}}(\text{sh}_{\mathbf{b}} \mathbf{a}_n, \text{sh}_{\mathbf{b}} \mathbf{c}_n) \leq 1$  which is impossible. It follows that  $C_{\mathbf{b},k} \cap \Delta^2 \tilde{T} = \emptyset$ .

Since  $C_{\mathbf{b},k}$  is a closed subset of  $\tilde{T} \times \tilde{T}$ , and  $\bar{\delta}_{\mathbf{b}}$  is a metric on  $\tilde{T}$ , there exists a constant  $\nu(\mathbf{b}) > 0$  such that  $\bar{\delta}_{\mathbf{b}}(\mathbf{a}, \mathbf{c}) \geq \nu(\mathbf{b})$  on  $C_{\mathbf{b},k}$ . Thus our statement holds for the set  $C_{\mathbf{b},k}$  of entourages separated by the fixed entourage  $\mathbf{b}$ .

We have  $A = G(\mathbf{a}_0)$ . If now  $\mathbf{a} - \mathbf{b} - \mathbf{c}(k)$  then  $\exists g \in G : \mathbf{b} = g(\mathbf{a}_0)$ , so  $g^{-1}\mathbf{a} - \mathbf{a}_0 - g^{-1}\mathbf{c}(k)$ . Thus  $\bar{\delta}_{\mathbf{b}}(\mathbf{a}, \mathbf{c}) = \bar{\delta}_{\mathbf{a}_0}(g^{-1}(\mathbf{a}), g^{-1}(\mathbf{c})) \geq \nu$ , where  $\nu = \nu(\mathbf{a}_0)$  is the above constant for  $\mathbf{a}_0$ . The Lemma is proved.  $\square$

The following Lemmas give a local description of  $C$ -quasigeodesics around tubes and horospheres.

**Lemma 5.2.** *There exists a constant  $D > 0$  such that for every  $C$ -quasigeodesic  $\gamma = \gamma(\mathbf{a}, \mathbf{c})$  in  $\mathcal{G}$  with the endpoints  $\mathbf{a}, \mathbf{c}$  we have :*

$$\forall \mathbf{b} \in \Psi_k(\mathbf{a}, \mathbf{c}) : d_A(\mathbf{b}, \gamma) \leq D, \quad (1)$$

where  $\Psi_k(\mathbf{a}, \mathbf{c}) = \{\mathbf{b} \in A : \mathbf{a} - \mathbf{b} - \mathbf{c}(k)\}$ .

*Proof:* By Lemma 5.1 we have  $\bar{\delta}_{\mathbf{b}}(\mathbf{a}, \mathbf{c}) \geq \nu$ , and so the Floyd length  $L_{f,\mathbf{b}}(\gamma)$  of  $\gamma$  is at least  $\nu$ . By Karlsson's Lemma 4.1 (see also 4.4.3) there exists a constant  $D > 0$  such that  $\gamma \cap B(\mathbf{b}, D) \neq \emptyset$  for the  $d_A$ -ball  $B(\mathbf{b}, D)$  in  $\mathcal{G}$  centered at  $\mathbf{b}$  with the radius  $D$ . The Lemma is proved.  $\square$

**Lemma 5.3.** *The following statements are true :*

- 1) *For any  $C > 0$  and  $E \geq 0$  there exists  $L > 0$  such that for any parabolic point  $p \in T$  and any  $C$ -quasigeodesic  $\gamma : [0, 1] \rightarrow \mathcal{G}$  one has*

$$d_A(\gamma(1), T(p)) \leq E \implies d_A(\gamma, \Pi_p(\gamma(0))) \leq L \quad (2).$$

- 2) There exists a constant  $D > 0$  such that for any parabolic point  $p \in T$  and any  $C$ -quasigeodesic  $\gamma : [0, \infty[ \rightarrow \mathcal{G}$  one has

$$\lim_{n \rightarrow \infty} \gamma(n) = p \implies d_A(\gamma, \Pi_p(\gamma(0))) \leq D. \quad (3)$$

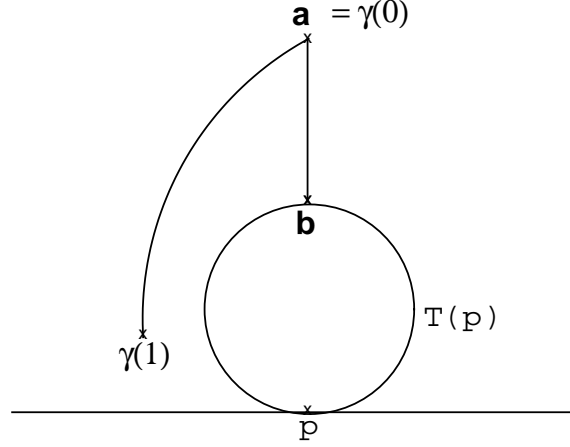


Figure 3: Quasigeodesics around horospheres.

*Proof:* 1) Suppose not, then there exist constants  $C$  and  $E$  such that for any  $n$  there exist a parabolic point  $p_n$  and a  $C$ -quasigeodesic  $\gamma_n : [0, 1] \rightarrow \mathcal{G}$  such that  $d_A(\gamma_n(1), T(p_n)) \leq E$  and  $d_A(\gamma_n, \Pi_{p_n}(\gamma_n(0))) > n$  for all  $n \in \mathbb{N}$ . By Lemma 3.17 there are at most finitely many  $G$ -non-equivalent parabolic points. So we may assume that  $p = p_n$  and let  $\mathbf{b}_n \in \Pi_p(\gamma_n(0))$ . By the same Lemma the group  $\text{Stab}_G p$  acts cofinitely on  $T(p)$  so we may also suppose that  $\mathbf{b} = \mathbf{b}_n$ .

Since  $d_A(\gamma_n(1), \mathbf{b})$  is unbounded the set  $\{\gamma_n(1)\}_n$  is infinite. As  $d_A(\gamma_n(1), T(p)) \leq E$  by Lemma 3.18 up to passing to a subsequence we have  $\gamma_n(1) \rightarrow p$ . Denote  $\mathbf{a}_n = \gamma_n(0)$  and  $\mathbf{c}_n = \gamma_n(1)$ . We have  $\forall n \ \mathbf{a}_n \neq \mathbf{b}$  so  $\mathbf{a}_n \notin T(p)$  and  $\mathbf{a}_n - \mathbf{b} - p$ . By Lemma 3.6 we obtain  $\mathbf{a}_n - \mathbf{b} - \mathbf{c}_n (n > n_0)$ . Thus Lemma 5.2 implies that  $d_A(\mathbf{b}, \gamma_n) \leq D$  which is a contradiction. The statement 1) is proved.

2) We have  $\lim_{n \rightarrow \infty} (\gamma(n) = \mathbf{c}_n) = p$  and without lost of generality we can suppose that  $\mathbf{a} = \gamma(0) \notin T(p)$ . Then arguing similarly we obtain  $\mathbf{a} - \mathbf{b} - \mathbf{c}_n (n > n_0)$  where  $\mathbf{b} = \Pi_p(\mathbf{a})$ . From Lemma 5.2 we have  $d_A(\mathbf{b}, \gamma) \leq D$ .  $\square$

The following Lemma is a generalization of the previous one to the geodesics with variable endpoints:

**Lemma 5.4.** *The following statements are true :*

- 1) For any  $C > 0$  and  $E \geq 0$  there exists  $M > 0$  such that for any parabolic point  $p \in T$  and any  $C$ -quasigeodesic  $\gamma : [-1, 1] \rightarrow \mathcal{G}$  one has

$$d_A(\{\gamma(-1), \gamma(1)\}, T(p)) \leq E \implies d_A(\gamma(0), \Pi_p(\gamma(0))) \leq M \quad (2').$$

- 2) There exists a constant  $D > 0$  such that for any parabolic point  $p \in T$  and any  $C$ -quasigeodesic  $\gamma : [-\infty, +\infty] \rightarrow \mathcal{G}$  one has

$$\lim_{n \rightarrow \pm\infty} \gamma(n) = p \implies d_A(\gamma(0), \Pi_p(\gamma(0))) \leq D \quad (3').$$

*Proof:* 1) As before using the finiteness of  $G$ -non-equivalent parabolic points, we fix a parabolic point  $p$ . Let  $\gamma_- = \gamma([-1, 0])$ , and  $\gamma_+ = \gamma([0, 1])$ . If  $\mathbf{a} = \gamma(0) \notin T(p)$  and  $\mathbf{b} = \Pi_p(\mathbf{a})$  then by the statement 1) of Lemma 5.3 we have  $d_A(\gamma_{\pm}, \mathbf{b}) \leq L$ . Let  $\mathbf{z} \in \gamma_+$  and  $\mathbf{y} \in \gamma_-$  be the points realizing these distances. Since there is a path from  $\mathbf{z}$  to  $\mathbf{y}$  through  $\mathbf{b}$  of length  $2L$ , the length  $l(\gamma(\mathbf{z}, \mathbf{y}))$  of the  $C$ -quasigeodesic  $\gamma(\mathbf{z}, \mathbf{y})$  between  $\mathbf{z}$  and  $\mathbf{y}$  is at most  $2L(C + 1)$ . So at least for one of these entourages, e.g.  $\mathbf{z}$ , we have  $l(\gamma(\mathbf{a}, \mathbf{z})) \leq L(C + 1)$ . By the triangle inequality we obtain  $d_A(\mathbf{a}, \mathbf{b}) \leq M = L(C + 2)$ .

The same argument and 5.3.2 imply the statement 2).  $\square$

The following Corollary establishes the uniform quasiconvexity of all horospheres and the quasiconvexity (simple) of every parabolic subgroup (see also [Ge1] and [GePo1]).

**Corollary 5.5.** *Suppose  $G$  acts 3-discontinuously and 2-cocompactly  $T$ . Then there exists  $M > 0$  such that for every  $p \in \mathcal{P}$  the horosphere  $T(p)$  is a  $M$ -quasiconvex subset of  $A$ .*

*Furthermore for every  $p \in \mathcal{P}$  there exists a constant  $D_p$  such that the parabolic subgroup  $H_p = \text{Stab}_G p$  is  $D_p$ -quasiconvex.*

*Proof:* Suppose first that  $\gamma \subset A$  is a  $C$ -quasigeodesic with  $\partial\gamma \subset T(p)$  for some  $p \in \mathcal{P}$ . By Lemma 5.4.1 for  $E = 0$  there exists a uniform constant  $M > 0$  such that  $\gamma \subset N_M(T(p))$ , where  $N_M(\cdot)$  denotes the  $M$ -neighborhood with respect to the distance  $d_A$ .

To prove the second part note that since  $G$  is finitely generated it is enough to prove it for the graph  $\mathcal{G}$  quasi-isometric to the Cayley graph. By Lemma 3.17 for every  $p \in \mathcal{P}$  the set  $T(p)/H_p$  is finite where  $H_p = \text{Stab}_G p$ . So there exists a constant  $E = E(p)$  such that  $H \subset N_E(T_p)$  and  $T(p) \subset N_E(H)$ . So if  $\gamma \subset A$  is a  $C$ -quasigeodesic with  $\partial\gamma \subset H$  then  $d_A(\partial\gamma, T(p)) \leq E$ . Then again by 5.4.1 there exists a constant  $M = M(p)$  such that  $\gamma \subset N_M(T(p))$ . So  $\gamma \subset N_{D_p}(H)$  where  $D_p = M + E$ .  $\square$

**Remark.** The above Lemmas 5.3 and 5.4 are close to some Lemmas contained in our work [GePo1] where the horospheres were defined without using the entourages. We need the above results in terms of entourages to apply them in the further argument where the language of entourages is crucial.

By Proposition 3.27.3b we have that for every  $d > 0$  there exists  $e = e(d) > 0$  such that

$$\forall p, q \in \mathcal{P} \text{ diam}(N_d(T(p)) \cap N_d(T(q))) \leq e. \quad (4)$$

**Definition 5.6.** Let  $\gamma \subset \tilde{T}$  be a  $C$ -quasigeodesic. We call an entourage  $\mathbf{v} \in \gamma$  *d-horospherical* if there exist parts  $[\mathbf{v}, \mathbf{c}]$  and  $[\mathbf{a}, \mathbf{v}]$  of  $\gamma$  of length greater than the constant  $e$  and which are contained in a  $d$ -neighborhood  $N_d(T(p))$  of a horosphere  $T(p)$ .

The entourage  $\mathbf{v} \in \gamma$  is called *non-horospherical* in the opposite case.

**Remark.** By (4) we can suppose that the parabolic point  $p$  with respect to which the (non)-horosphericity is considered is unique.

**Lemma 5.7.** Let  $\gamma = \gamma(\mathbf{a}, \mathbf{c})$  be a  $c$ -quasigeodesic. Suppose that  $P = P(\mathbf{a}, \mathbf{c})$  is a non-refinable tube having the same ending vertices  $\mathbf{a}$  and  $\mathbf{c}$  as  $\gamma$ . For every sufficiently large  $d > 0$  there exists a constant  $E > 0$  such that  $d_A(\mathbf{g}, P) \leq E$  for every  $d$ -non-horospherical point  $\mathbf{g} \in \gamma$ .

*Proof:* Note that the non-refinable tube  $P(\mathbf{a}, \mathbf{c})$  exists by Proposition 3.25. By Lemma 5.2 there exists  $D > 0$  such that for every  $\mathbf{p}_i \in P$  we have  $d_A(\mathbf{p}_i, \gamma) \leq D$  ( $i = 1, \dots, m$ ). So let us fix a non-horospherical entourage  $\mathbf{g} \in \gamma$ , and let  $\mathbf{g}_i \in \gamma$  be such that  $d_A(\mathbf{p}_i, \gamma) = d_A(\mathbf{p}_i, \mathbf{g}_i)$  ( $i = 0, \dots, m$ ). Let us also assume that  $\mathbf{g} \in \gamma(\mathbf{g}_i, \mathbf{g}_{i+1})$  where  $\gamma(\mathbf{g}_i, \mathbf{g}_{i+1})$  denotes the part of  $\gamma$  between  $\mathbf{g}_i$  and  $\mathbf{g}_{i+1}$ .

By Corollary 3.29 there exists a constant  $C > 0$  such that if  $d_A(\mathbf{p}_i, \mathbf{p}_{i+1}) > C$  then the pair  $\{\mathbf{p}_i, \mathbf{p}_{i+1}\}$  is contained in a horosphere  $T(p)$ . In this case  $\{\mathbf{g}_i, \mathbf{g}_{i+1}\} \subset N_D(T(p))$  and by Lemma 5.4 we have that  $\gamma(\mathbf{g}_i, \mathbf{g}_{i+1}) \subset N_L(T(p))$  for some  $L = L(D) > 0$ . Let  $d$  be any number bigger than  $L$ . If  $\mathbf{g}$  is  $d$ -non-horospherical then by 5.6  $d_A(\mathbf{g}, \mathbf{g}_i)$  or  $d_A(\mathbf{g}, \mathbf{g}_{i+1})$  is less than  $e$ . Thus  $d_A(\mathbf{g}, P) \leq e + d$ .

If now  $d_A(\mathbf{p}_i, \mathbf{p}_{i+1}) \leq C$  then  $d_A(\mathbf{g}_i, \mathbf{g}_{i+1}) \leq c(C + 2D) + c$ . So  $d_A(\mathbf{g}, P) \leq d_A(\mathbf{g}_i, \mathbf{g}) + D \leq c(C + 2D) + c + D$ .

Put  $E = \max\{e + d, c(C + 2D) + c + D\}$ . The Lemma is proved.  $\square$

**Remark.** The constants  $d$  and  $e$  depend on the constants  $D$ ,  $C$  and  $L = L(D)$  given respectively by the statements 5.2, 3.29 and 5.4.

## 6 Tight curves in $\mathcal{G}$ .

Let a finitely generated group  $G$  act 3-discontinuously and 2-cocompactly on a compactum  $T$ . For a parabolic point  $p$  we denote by  $N(T(p))$  a neighborhood of the horosphere  $T(p)$  in the graph  $\mathcal{G}$  (see Section 3.2). The notation  $\text{diam}(\cdot)$  is used for the diameter of a set with respect to the distance  $d_A$  and  $|\cdot|$  stands for the length of a curve. We denote by  $c^{-1}(n)$  the linear function  $\frac{n}{c} - c$  for some constant  $c > 0$ .

**Definition 6.1.** For positive integers  $l$  and  $c$ , a curve  $\gamma : I \rightarrow \mathcal{G}$  is called  $(l, c)$ -tight (or just *tight* when the values of  $l$  and  $c$  are fixed) if for every  $J \subset I$  the following conditions hold:

1.  $|J| \leq l \implies \gamma|_J$  is a  $c$ -quasigeodesic.

2. If  $|\gamma(J) \cap N(T(p))| > l$  for some  $p \in \mathcal{P}$  then  $\text{diam}(\gamma(\partial J)) > c^{-1}(l)$ .  $\square$

The rest of the Section is devoted to the proof of the following Theorem describing the non-horospherical points (see Definition 5.6) of tight curves.

**Theorem B.** *For every  $c > 0$  and  $d > 0$  there exist positive constants  $l_0, w_0, c_0$  such that for all  $l \geq l_0$  and every  $(l, c)$ -tight curve  $\gamma \subset \mathcal{G}$  there exists a  $c_0$ -quasigeodesic  $\alpha \subset A$  such that every  $d$ -non-horospherical vertex of  $\gamma$  belongs to the  $w_0$ -neighborhood  $N_{w_0}(\alpha)$  of  $\alpha$ .*

The following three lemmas are close to the results of the previous Section. We use below the notation  $\text{diam}_{\bar{\delta}_{\mathbf{v}}}$  for the diameter of a set with respect to the shortcut metric  $\bar{\delta}_{\mathbf{v}}$  ( $\mathbf{v} \in A$ ) on  $\tilde{T}$  (see Lemma 4.3)

**Lemma 6.2.** *There exist positive constants  $\rho$  and  $d$  such that for every  $c$ -quasigeodesic  $\gamma : I \rightarrow \mathcal{G}$  of non-zero length and a  $d$ -non-horospherical point  $\gamma(0) \in \mathcal{G}$  one has:*

$$\text{diam}_{\bar{\delta}_{\gamma(0)}}(\gamma(\partial I)) > \rho.$$

*Proof:* Let us first prove that there exists a constant  $r > 0$  such that for some  $\rho = \rho(r)$  we have

$$d_A(\gamma(0), \gamma(\partial I)) > r \implies \bar{\delta}_{\gamma(0)}(\gamma(\partial I)) > \rho \quad (*).$$

Suppose not. Then for every  $d > 0$  there exists a sequence of quasigeodesics  $\gamma_n$  such that  $d_A(\gamma_n(0), \gamma_n(\partial I)) \rightarrow +\infty$  and  $\bar{\delta}_{\gamma_n(0)}(\gamma_n(\partial I)) \rightarrow 0$  where  $\gamma_n(0)$  is a  $d$ -non-horospherical point of  $\gamma_n$ .

Up to choosing a subsequence we may suppose that the sequence  $(\gamma_n)_n$  converges in the Tikhonov topology to a  $c$ -quasigeodesic  $\gamma : \mathbb{Z} \rightarrow \mathcal{G}$  such that  $\lim_{n \rightarrow \pm\infty} \gamma(n) = p \in T$ . Then  $\gamma$  is a horocycle at  $p$  and by [GePo1, Lemma 3.6] the point  $p$  is parabolic. By Lemma 5.4.2 for every  $i \in \mathbb{Z}$  the distance  $d_A(\gamma_n(i), T(p))$  is uniformly bounded by a constant  $D > 0$ . So the points  $\gamma_n(0)$  are  $D$ -horospherical for sufficiently large  $n$ . The obtained contradiction proves (\*).

We are left now with the case when  $d_A(\gamma(0), \gamma(\partial I)) \leq r$  where the constant  $r$  satisfies (\*). Suppose first that the distance between  $\gamma(0)$  and both endpoints of  $\gamma(\partial I)$  is less than  $r$ . By translating  $\gamma(0)$  to a fixed basepoint  $\mathbf{v} \in A$  we obtain that  $\gamma$  is contained in a finite ball  $B(\mathbf{v}, r + c(r))$ . Then the  $\bar{\delta}_{\mathbf{v}}$ -length of  $\gamma$  is uniformly bounded from below. If the distance between  $\gamma(0)$  and only one of its endpoints is bigger than  $r$  then the  $\bar{\delta}$ -length of  $\gamma$  is still bounded from below.

Denoting by  $\rho$  the minimum among all of these constants we obtain the Lemma.  $\square$

**Remark.** Above we have used Lemma 3.6 from [GePo1] stated there for the Cayley graphs. Since our graph  $\mathcal{G}$  is quasi-isometric to the Cayley graph this result can be applied.

Recall that  $A = G(\mathbf{a}_0)$  is the vertex set of the graph  $\mathcal{G}$ . Using a "refining" procedure we will now introduce a new graph  $\mathcal{G}^*$  whose vertex set  $A^*$  satisfies some additional conditions.

From now on we fix the constant  $d$  and  $\rho = \rho(d)$  coming from Lemma 6.2 and an integer  $k > 3$  which will be used in the betweenness relation below. Let  $\delta$  be a number such that

$$0 < \delta < \frac{\rho}{k+2}. \quad (**)$$

Definition of the set  $A^*$ : For every  $\mathbf{v} \in A$  denote by  $\mathbf{v}^*$  the entourage  $\{\{x, y\} \in \mathbf{S}^2 T : \bar{\delta}_{\mathbf{v}}(x, y) < \delta\}$ .

It follows from the following Lemma that the compactifying topology on  $T$  coming from the graphs  $A^*$  and  $A$  is the same.

**Lemma 6.3.**  $\forall p \in T$   $\mathbf{a}_n \rightarrow p$  if and only if  $\mathbf{a}_n^* \rightarrow p$ .

*Proof:* Suppose first that  $\mathbf{a}_n \rightarrow p$  and  $\mathbf{a}_n^* \not\rightarrow p$ . Then there exists a neighborhood  $U_p$  of the point  $p$  such that  $U_p'$  is not  $\mathbf{a}_n^*$ -small for  $n > n_0$ . So  $\exists \mathbf{x}_n, \mathbf{y}_n \in U_p' : \bar{\delta}_{\mathbf{a}_n}(\mathbf{x}_n, \mathbf{y}_n) > \delta$ . It follows that up to subsequences we have  $\mathbf{x}_n \rightarrow x \in T$ ,  $\mathbf{y}_n \rightarrow y \in T$  ( $n \rightarrow \infty$ ) and  $x \neq y \neq p \neq x$ . Let  $U_x$  and  $U_y$  be closed neighborhoods of  $x$  and  $y$  such that  $U_p \cap U_x \cap U_y = \emptyset$ .

Let  $H(U_{x,y}) \subset \mathcal{G}$  denote the set of geodesics whose endpoints are situated in  $U_{x,y} = U_x \sqcup U_y$ . By [GePo1, Main Lemma]  $\overline{H(U_{x,y})} \cap T = \overline{U_{x,y}} \cap T$  where  $\overline{U_{x,y}}$  means the closure of  $U_{x,y}$  in  $\tilde{T} = A \sqcup \text{Ent } T$ . It follows that the geodesics  $\gamma_n(\mathbf{x}_n, \mathbf{y}_n) \subset \mathcal{G}$  between  $\mathbf{x}_n$  and  $\mathbf{y}_n$  do not intersect a neighborhood  $V_p \subset U_p$  of  $p$  ( $n > n_0$ ). Since  $\mathbf{a}_n \rightarrow p$  we have  $d_A(\mathbf{a}_n, \gamma_n) \rightarrow \infty$ . By Karlsson Lemma 4.1 (see also Remark 4.4.3) we obtain that  $\bar{\delta}_{\mathbf{a}_n}(\mathbf{x}_n, \mathbf{y}_n) < \delta$  ( $n > n_0$ ) which is a contradiction.

Suppose now  $\mathbf{a}_n^* \rightarrow p$  and  $\mathbf{a}_n \not\rightarrow p$ . Then up to a subsequence we have  $\mathbf{a}_n \rightarrow q \neq p$ . Let  $U_p$  be a neighborhood of  $p$  such that  $U_p'$  is  $\mathbf{a}_n^*$ -small ( $n > n_0$ ). We have  $d_A(\mathbf{a}_n, U_p) \rightarrow +\infty$ . Then by Karlsson Lemma  $\forall x, y \in U_p$   $\bar{\delta}_{\mathbf{a}_n}(x, y) < \delta$  ( $n > n_0$ ). So  $U_p$  and  $U_p'$  are both  $\mathbf{a}_n^*$ -small ( $n > n_0$ ) which is impossible (see (1) of 3.1).  $\square$

The need of the graph  $A^*$  is explained by the following:

**Lemma 6.4.** *There exists constant  $w > 0$  such that for every quasigeodesic  $\gamma : I \rightarrow \mathcal{G}$  containing three vertices  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A$  the following is true:*

$$\mathbf{b} \text{ is } d\text{-non-horospherical} \wedge d_A(\mathbf{b}, \{\mathbf{a}, \mathbf{c}\}) > w \implies \mathbf{a}^* - \mathbf{b}^* - \mathbf{c}^*(k).$$

*Proof:* Suppose not and there are sequences  $\mathbf{a}_n, \mathbf{c}_n$  and  $\mathbf{b}_n$  such that  $\mathbf{b}_n$  is  $d$ -non-horospherical,  $d_A(\mathbf{b}_n, \{\mathbf{a}_n, \mathbf{c}_n\}) \rightarrow \infty$  and  $\mathbf{a}_n^* - \mathbf{b}_n^* - \mathbf{c}_n^*(k)$  is not true. Since  $A$  is  $G$ -finite we can suppose that  $\mathbf{b}_n = \mathbf{b}$ . Up to a subsequence we have  $\mathbf{a}_n \rightarrow p$ ,  $\mathbf{c}_n \rightarrow q$ . Let  $\gamma_n = \gamma_n(\mathbf{a}_n, \mathbf{c}_n) \subset \mathcal{G}$  be a geodesic between  $\mathbf{a}_n$  and  $\mathbf{c}_n$ . Since  $\mathbf{b}$  is non-horospherical we have by Lemma 6.2 that  $\bar{\delta}_{\mathbf{b}}(p, q) > \rho$ , hence  $p \neq q$ .

Let  $U_p$  and  $U_q$  be disjoint  $\mathbf{b}^*$ -small neighborhoods of  $p$  and  $q$  respectively. So  $\bar{\delta}_{\mathbf{b}}(U, V) > \rho - 2\delta$ , and (\*\*) yields  $\bar{\delta}_{\mathbf{b}}(U_p, U_q) \geq \rho - 2\delta > k \cdot \delta$ . We obtain  $\Delta_{\mathbf{b}^*}(U_p, U_q) > k$ . By Lemma 6.3 we also have  $\mathbf{a}_n^* \rightarrow p$  and  $\mathbf{c}_n^* \rightarrow q$ . So  $U_p'$  and  $U_q'$  are  $\mathbf{a}_n^*$ -small and  $\mathbf{c}_n^*$ -small respectively ( $n > n_0$ ). Hence  $U_p \supset \text{sh}_{\mathbf{b}^*} \mathbf{a}_n^*$  and  $U_q \supset \text{sh}_{\mathbf{b}^*} \mathbf{c}_n^*$ . It follows that  $\Delta_{\mathbf{b}^*}(\text{sh}_{\mathbf{b}^*} \mathbf{a}_n^*, \text{sh}_{\mathbf{b}^*} \mathbf{c}_n^*) \geq (\Delta_{\mathbf{b}^*}(U_p, U_q) > k$ . Therefore  $\mathbf{a}_n^* - \mathbf{b}^* - \mathbf{c}_n^*(k)$  which is a contradiction.  $\square$

**Lemma 6.5.** *For every  $d > 0$  there exists a constant  $l_0$  such that for every parabolic point  $p$ , and all entourages  $\mathbf{b}, \mathbf{c}, \mathbf{d} \in N_d(T(p))$ , and  $\mathbf{a} \in A$  one has*

$$\forall l > l_0 : d_A(\mathbf{b}, \mathbf{c}) > l \wedge d_A(\mathbf{b}, \mathbf{d}) > l \wedge \mathbf{a}^* - \mathbf{b}^* - \mathbf{c}^*(k) \implies \mathbf{a}^* - \mathbf{b}^* - \mathbf{d}^*(k-1) \quad (1)$$

*Proof:* Since by Lemma 3.17 the set of parabolic points is  $G$ -finite it is enough to prove the statement for a fixed parabolic point  $p \in T$ . By Lemma 3.18 the parabolic point  $p$  is the unique limit point of  $N_d(T(p))$ . By definition of the topology on  $T \sqcup \text{Ent}T$  for sufficiently large  $l_0$  our assumption implies that the entourages  $\mathbf{c}$  and  $\mathbf{d}$  are sufficiently close to  $p$ . By Lemma 6.3 the entourages  $\mathbf{c}^*$  and  $\mathbf{d}^*$  are also close to  $p$ . So for every  $\mathbf{b}^*$ -small neighborhood  $U_p$  of  $p$  its complement  $U'_p$  is  $\mathbf{c}^*$ -small and  $\mathbf{d}^*$ -small for  $l > l_0$ . Then  $\text{sh}_{\mathbf{b}^*}\mathbf{c}^* \subset U_p$  and  $\text{sh}_{\mathbf{b}^*}\mathbf{d}^* \subset U_p$ . Therefore  $\tilde{\Delta}_{\mathbf{b}^*}(\text{sh}_{\mathbf{b}^*}\mathbf{c}^*, \text{sh}_{\mathbf{b}^*}\mathbf{d}^*) \leq 1$ . We obtain  $\Delta_{\mathbf{b}^*}(\text{sh}_{\mathbf{b}^*}\mathbf{d}^*, \text{sh}_{\mathbf{b}^*}\mathbf{a}^*) \geq \Delta_{\mathbf{b}^*}(\text{sh}_{\mathbf{b}^*}\mathbf{a}^*, \text{sh}_{\mathbf{b}^*}\mathbf{c}^*) - \tilde{\Delta}_{\mathbf{b}^*}(\text{sh}_{\mathbf{b}^*}\mathbf{c}^*, \text{sh}_{\mathbf{b}^*}\mathbf{d}^*) > k - 1$ .  $\square$

**Remark 6.6.** (*about the constants*). Since now on we assume that the tightness constant  $l$  is much bigger than the horosphericity constants  $d, e = e(d)$  (see Definition 5.6 and the Remark after it) and  $w$  (see 6.4). We will also suppose that the chosen constants satisfy the following relations:

$$l_0 > 4w, \quad w > e.$$

*Proof of Theorem B.* Recall that for a fixed constant  $d > 0$  by Lemma 6.2 we have found  $\rho = \rho(d)$  and have defined the set  $A^*$  of vertices of a new graph of entourages. Since now on the term "(non)-horosphericity" will mean " $d$ -(non)-horosphericity".

Before going into the details we outline the proof of the theorem. We start by choosing non-horospheric points  $\mathbf{v}_n$  of the curve  $\gamma$  which give by Lemma 6.4 an auxiliary tube  $P^* = \{\mathbf{v}_n^*\}$  in the graph  $A^*$ . There is a quasi-geodesic  $\alpha^* \subset A^*$  whose non-horospheric points are in a bounded distance from  $P^*$  (Lemma 5.7). Since the graphs  $\mathcal{G}$  and  $\mathcal{G}^*$  are  $G$ -finite the map  $\varphi : \mathbf{v} \rightarrow \mathbf{v}^*$  is a quasi-isometry between them. This will give us a quasi-geodesic  $\alpha \subset A$  satisfying the statement of the Theorem. All the remaining constants will be found in the course of the proof.

To construct the tube  $P^*$  we proceed inductively by choosing vertices of  $\gamma$  as follows. Let  $\gamma(0)$  be the first non-horospheric point on  $\gamma$ , then we put  $\mathbf{v}_0^* = \gamma^*(0)$ . Suppose that a point  $\mathbf{v}_n^* = \gamma^*(n)$  is already chosen. Then for the constant  $w$  fixed above we choose  $i_{n+1} \geq i_n + w$  such that  $\gamma(i_{n+1})$  is the first non-horospheric point on  $\gamma$  after  $\gamma(i_n + w)$ . We set  $\mathbf{v}_{n+1}^* = \gamma^*(i_{n+1})$ . The following Proposition shows that for every  $n$  each three chosen neighboring vertices form a tube  $\mathbf{v}_{n-1}^* - \mathbf{v}_n^* - \mathbf{v}_{n+1}^* (k-2)$  for the integer  $k$  fixed above. Then all the constructed vertices will give a tube  $P^* = \mathbf{v}_0^* - \mathbf{v}_1^* - \dots - \mathbf{v}_m^* (k-2)$ .

**Proposition 6.7.** *For every  $n \in \mathbb{N}$  one has  $\mathbf{v}_{n-1}^* - \mathbf{v}_n^* - \mathbf{v}_{n+1}^* (k-2)$ .*

*Proof of the Proposition.* There are four different cases depending on the lengths  $|\gamma|_{[i_n, i_{n+1}]} = i_{n+1} - i_n$  of the parts of  $\gamma$  ( $n \in \mathbb{N}$ ).

*Case 1.*  $i_n - i_{n-1} \leq l/2 \wedge i_{n+1} - i_n \leq l/2$ ,

By definition of a tight curve the points  $\gamma(i_{n-1}), \gamma(i_n), \gamma(i_{n+1})$  belong to a  $c$ -quasigeodesic part of  $\gamma$  so the result follows from Lemma 6.4.

*Case 2.*  $i_{n+1} - i_{n-1} > l$ .

There are three subcases.

*Subcase 2.1.*  $i_n - i_{n-1} \leq l/2 \wedge i_{n+1} - i_n > l/2$ ,



Since  $\gamma(i_{n+1})$  is the first non-horospherical point on  $\gamma$  after  $\gamma(i_n + w)$  and  $w < l/2$  the point  $\gamma(i_n + w)$  is horospherical. Since  $w > e$  by the Remark after 5.6 there exists a unique horosphere  $T(p)$  such that  $d_A(\gamma(i_n + w), T(p)) \leq d$ . As  $\gamma|_{[i_n, i_n + w]}$  is a  $c$ -quasigeodesic we have

$$d_A(\gamma(i_n), T(p)) < cw + c + d. \quad (***)$$

Furthermore Lemma 6.4 yields:

$$\gamma^*(i_{n-1}) - \gamma^*(i_n) - \gamma^*(i_{n-1} + l) \quad (k). \quad (2)$$

Since  $i_{n+1} - i_{n-1} > l$  the point  $\gamma(i_{n-1} + l)$  is also horospherical and  $\gamma(i_{n-1} + l) \in \gamma|_{[i_n + w, i_{n+1}]}$ . The curve  $\gamma|_{[i_n, i_n + l]}$  is still  $c$ -quasigeodesic so we have

$$d_A(\gamma(i_n), \gamma(i_{n-1} + l)) > \frac{i_{n-1} + l - i_n}{c} - c \geq \frac{l}{2c} - c > \frac{l}{4c}, \quad (3)$$

where we assume that  $l > l_0 > 4c^2$  for the constant  $l_0$  from Lemma 6.5.

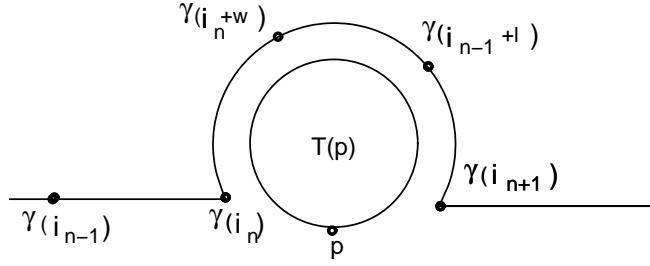


Figure 3: Tight curves around horospheres.

By construction we can also suppose that  $\gamma(i_{n+1}) \in N_d(T(p))$  for the  $d$ -neighborhood  $N_d(T(p))$  of  $p$ . Indeed otherwise there would exist another non-horospherical point on  $\gamma$  after  $\gamma(i_n + w)$  and preceding  $\gamma(i_{n+1})$ . So by  $(***)$   $\{\gamma(i_n), \gamma(i_{n+1})\} \subset N_{d_0}(T(p))$ , where  $d_0 = cw + c + d$ .

If, first,  $i_{n+1} - i_n \leq l$  then  $\gamma|_{[i_n, i_{n+1}]}$  is a  $c$ -quasigeodesic, and  $d_A(\gamma(i_n), \gamma(i_{n+1})) > l/2c - c > l/4c$ . Hence by the choice of  $l_0$  (see Remark 6.6) and all  $l > l_0$  we have from (2), (3) and Lemma 6.5

$$\gamma^*(i_{n-1}) - \gamma^*(i_n) - \gamma^*(i_{n+1}) \quad (k-1). \quad (4)$$

If now  $i_{n+1} - i_n > l$  then applying 6.1.2 to  $N_{d_0}(p)$  we obtain  $d_A(\gamma(i_n), \gamma(i_{n+1})) > c^{-1}(l)$  and again (4) follows from (2), (3) and Lemma 6.5.  $\square$

*Subcase 2.2.*  $i_n - i_{n-1} \geq l/2 \wedge i_{n+1} - i_n \leq l/2$ ,

The argument is similar to that of Subcase 2.1 but it works in the opposite direction. We have the tube  $\gamma^*(i_{n+1}) - \gamma^*(i_n) - \gamma^*(i_{n+1} - l) \quad (k)$ . As above if  $i_n - i_{n-1} \leq l$  then the curve  $\gamma|_{[i_{n-1}, i_n]}$  is  $c$ -quasigeodesic and so its diameter is greater than  $l/4c$ . If not then using the tightness property of it, we obtain that  $d_A(\gamma(i_{n-1}), \gamma(i_n)) > c^{-1}(l)$  and (4) follows by the same argument as in Subcase 2.1.

*Subcase 2.3.*  $i_n - i_{n-1} \geq l/2 \wedge i_{n+1} - i_n \geq l/2$ ,

In this case we have that the points  $\gamma(i_n - l/4)$  and  $\gamma(i_n + l/4)$  preceding respectively  $\gamma(i_n)$  and  $\gamma(i_{n+1})$  are both horospherical. Indeed  $w < l/4$  and  $\gamma(i_n)$  and  $\gamma(i_{n+1})$  are the first non-horospherical points after  $\gamma(i_{n-1})$  and  $\gamma(i_n)$  respectively. So we can suppose that  $\gamma(i_n) \in N_d(T(p))$  and  $\gamma(i_{n+1}) \in N_d(T(q))$  where  $p$  and  $q$  are distinct parabolic points. Since  $\gamma|_{[i_n-l/4, i_n+l/4]}$  is a quasigeodesic by Lemma 6.4 we have

$$\gamma^*(i_n - l/4) - \gamma^*(i_n) - \gamma^*(i_n + l/4) (k). \quad (5)$$

We also have  $d_A(\gamma(i_{n-1}), \gamma(i_n))$  and  $d_A(\gamma(i_n), \gamma(i_{n+1}))$  are both greater than  $l/4c$ . Indeed if  $i_n - i_{n-1} > l$  then by  $(l, c)$ -tightness we have  $d_A(\gamma(i_{n-1}), \gamma(i_n)) > c^{-l}(l) > l/4c$ . If  $i_n - i_{n-1} \leq l$  then  $\gamma|_{[i_{n-1}, i_n]}$  is  $c$ -quasigeodesic, and as above  $d_A(\gamma(i_{n-1}), \gamma(i_n)) > l/4c$ . In the same way we obtain  $d_A(\gamma(i_n), \gamma(i_{n+1})) > l/4c$ .

Applying now Lemma 6.5 to (5) two times for  $l > 4cl_0$  we obtain

$$\gamma^*(i_{n-1}) - \gamma^*(i_n) - \gamma^*(i_{n+1}) (k - 2).$$

The Proposition is proved.  $\square$

We continue the proof of Theorem B. By Proposition 6.7 the curve  $\gamma$  admits a set of non-horospherical points  $\mathbf{v}_n = \gamma(i_n)$  such that  $\mathbf{v}_n^* = \varphi(\gamma(i_n))$  is a vertex of the tube  $P^*$ . Let  $\mathbf{u} = \gamma(i)$  be a non-horospherical point of  $\gamma$  which does not belong to the set  $\{\mathbf{v}_n\}_n$ . Then by construction  $i_n \leq i < i_n + w$  for some  $i_n \in \{0, \dots, m\}$ . Since  $w < l$  the curve  $\gamma|_{[i_n, i_n+w]}$  is a  $c$ -quasigeodesic so  $d_A(\mathbf{v}_n, \mathbf{u}) \leq cw + c$ . The map  $\varphi : \mathbf{u} \in A \rightarrow \mathbf{u}^* \in A^*$  is a quasi-isometry so  $d_{A^*}(\mathbf{u}^*, \mathbf{v}_n^*) \leq w_1$  for some uniform constant  $w_1 > 0$ . Let  $\alpha^*$  be a geodesic in the graph  $\mathcal{G}^*$  with the same endpoints as  $P^*$ . Then by Lemma 5.2 (applied to the graph  $\mathcal{G}^*$ ) there is a constant  $D^* > 0$  such that  $\forall \mathbf{v}^* \in P^* : d_{A^*}(\alpha^*, \mathbf{v}^*) \leq D^*$ . So for every non-horospherical point  $\mathbf{u} \in \gamma$  we have  $d_{A^*}(\mathbf{u}^*, \alpha^*) \leq d_{A^*}(\mathbf{u}^*, \mathbf{v}^*) + d_{A^*}(\mathbf{v}^*, \alpha^*) \leq w_1 + D^*$  where  $\mathbf{v}^* \in P^*$ . The map  $\varphi^{-1} : \mathbf{u}^* \rightarrow \mathbf{u}$  is a quasi-isometry too. Hence  $\alpha = \varphi^{-1}(\alpha^*)$  is a  $c_0$ -quasi-geodesic in  $\mathcal{G}$  such that for every non-horospherical point  $\mathbf{u} \in \gamma$  we have  $d_A(\mathbf{u}, \alpha) \leq w_0$  for some positive constants  $c_0$  and  $w_0$ . Theorem B is proved.  $\square$ .

## 7 Floyd quasiconvexity of parabolic subgroups.

Let  $G \curvearrowright T$  be a 3-discontinuous and 2-cocompact action of a finitely generated group  $G$  on a compactum  $T$ . Let  $\Gamma$  be a locally finite, connected graph on which  $G$  acts discontinuously and cofinitely (e.g. its Cayley graph or the graph of entourages). We denote by  $d(\cdot, \cdot)$  the graph distance of  $\Gamma$ . Let  $f : \mathbb{N} \rightarrow \mathbb{R}_{>0}$  be a scaling function esatisfying the following conditions (1-2) (see Section 4):

$$\exists \lambda > 0 \forall n \in \mathbb{N} : 1 < \frac{f(n)}{f(n+1)} < \lambda \quad (1)$$

$$\sum_{n \in \mathbb{N}} f(n) < +\infty. \quad (2)$$

To precise that  $f$  satisfies (1) with respect to some  $\lambda \in ]1, \infty[$  we will say that the function  $f$  is  $\lambda$ -slow. Denote by  $\delta_f$  the corresponding Floyd metric on  $\Gamma$  with respect to a fixed vertex  $v \in \Gamma^0$ .

By a standard argument based on Arzela-Ascoli theorem it follows that the Floyd completion  $\bar{\Gamma}_f$  of the graph  $\Gamma$  is a geodesic (strictly intrinsic) space (see e.g. [BBI, Theorem 2.5.14]. We call Floyd geodesic (or  $\delta_f$ -geodesic) a geodesic in the space  $\bar{\Gamma}_f$  with respect to the Floyd  $\delta_f$ -metric. The geodesics in  $\Gamma$  with respect to the graph distance  $d$  we call below  $(d-)$ geodesics.

The set  $\Gamma^0/G = K$  is finite so we can identify in  $\Gamma$  a subgroup  $H$  of  $G$  with the orbit  $HK = \bigcup_{h \in H} hK \subset \Gamma^0$ . Let  $N_R(H)$  denote the  $R$ -neighborhood of  $HK$  in  $\Gamma$  for the graph metric.

**Definition 7.1.** Let  $\Gamma$  be a locally finite, connected graph possessing a  $G$ -finite action. A subgroup  $H$  of  $G$  is called *Floyd quasiconvex* in  $\Gamma$  if there exists a constant  $R = R(H) > 0$  such that every Floyd geodesic  $\gamma = \gamma(h_1, h_2) \subset \Gamma$  for the metric  $\delta_f$  having the endpoints  $h_i$  in  $H$  belongs to  $N_R(H)$ :  $\forall x \in \gamma : d(x, H) < R$ .

By Corollary 5.5 every parabolic subgroup of  $G$  is quasiconvex with respect to the word metric (see also [Ge1]). The aim of this Section is to prove the following Theorem stating the Floyd quasiconvexity of parabolic subgroups.

**Theorem C.** Let  $G$  be a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum  $T$ . Let  $\Gamma$  be a locally finite, connected graph admitting a cocompact discontinuous action of  $G$ . Then there exists a constant  $\lambda_0 \in ]1, \infty[$  such that for every  $\lambda \in ]1, \lambda_0[$  and every  $\lambda$ -slow Floyd scaling function  $f$  satisfying (1-2), each parabolic subgroup  $H$  of  $G$  is Floyd quasiconvex for the Floyd metric  $\delta_f$ .  $\square$

We start with two Lemmas.

**Lemma 7.2.** *For every  $r > 0$  there exists  $\lambda_0 > 1$  such that  $\forall \lambda \in ]1, \lambda_0[$  and every  $\lambda$ -slow function  $f$  the condition  $d(x, y) \leq r$  ( $x, y \in \Gamma^0$ ) implies that every Floyd  $\delta_f$ -geodesic  $\gamma = \gamma(x, y) \subset \Gamma$  whose endpoints are  $x$  and  $y$  is a geodesic in  $\Gamma$ .*

**Remark.** A similar statement for  $\delta$ -hyperbolic spaces is proved in [Gr, Lemma 7.2.1]

*Proof:* Let  $v \in \Gamma^0$  be a basepoint and let  $R = d(v, \{x, y\})$ . Denote by  $\omega = \omega(x, y)$  a  $\Gamma$ -geodesic between  $x$  and  $y$  for which  $|\omega| = r$ . We have  $L_f(\omega) = \sum_{i=1}^l f(d(v, \{x_i, x_{i+1}\})) \leq rf(|R - r|)$ .

Suppose by contradiction that  $\gamma$  is not  $d$ -geodesic and so  $|\gamma| \geq r + 1$ . Let  $\gamma'$  be the part of  $\gamma$  in the ball  $B(v, R + r + 1)$  of radius  $R + r + 1$  centered at  $v$ . By the triangle inequality we also have  $|\gamma'| \geq r + 1$ . So  $L_f(\gamma) \geq L_f(\gamma') \geq (r + 1)f(R + r + 1)$ . Hence

$$\frac{f(R + r + 1)}{f(|R - r|)} \leq \frac{L_f(\gamma)}{(r + 1)f(|R - r|)} \leq \frac{L_f(\omega)}{(r + 1)f(|R - r|)} \leq \frac{r}{r + 1}.$$

Since  $f$  is  $\lambda$ -slow we have  $\frac{f(r + R + 1)}{f(|R - r|)} > \frac{1}{\lambda^{2r+1}}$ . Thus

$$\frac{1}{\lambda^{2r+1}} < \frac{r}{r + 1}. \quad (*)$$

Then there exists  $\lambda_0 > 1$  such that for  $\lambda \in ]1, \lambda_0[$  the inequality (\*) is not true for a fixed  $r > 0$ . So for such  $\lambda_0$  we have a contradiction. The Lemma is proved.  $\square$

**Remark.** Obviously if  $r$  is not fixed and tends to infinity the above constant  $\lambda_0$  does not exist.

The group  $G$  acts discontinuously and cofinitely on the graph  $\Gamma$  and on the graph  $\mathcal{G}$  of entourages (see Section 3). Since the set  $\Gamma^0/G = K$  is finite and  $\mathcal{G}^0/G = \{\mathbf{a}_0\}$  ( $\mathbf{a}_0 \in A$ ) the correspondence  $K \rightarrow \mathbf{a}_0$  extends  $G$ -equivariantly to the quasi-isometry  $\psi : gK \rightarrow g\mathbf{a}_0$  ( $g \in G$ ). In the same way we define the inverse quasi-isometric map  $\psi^{-1} : \mathcal{G} \rightarrow \Gamma$  for which  $\psi^{-1}(\mathbf{a}_0) \in K$ .

For a parabolic point  $p \in \mathcal{P}$  let  $H$  denote the stabilizer of  $p$  in  $G$ .

**Lemma 7.3.** *The map  $\psi$  extends continuously by the identity map to the map  $\Gamma \sqcup \mathcal{P} \rightarrow \mathcal{G} \sqcup \mathcal{P}$ . Furthermore for any  $d > 0$  there exists  $d' = d'(d, p)$  such that  $\psi(N_d(H))$  belongs to a  $d'$ -neighborhood  $N_{d'}(T(p))$  of the horosphere  $T(p) \subset \mathcal{G}$ ; and vice versa  $\psi^{-1}(N_d(T(p))) \subset N_{d'}(H)$ .*

*Proof:* It follows from [GePo1, Lemma 3.8] that the unique limit point of  $N_d(H)$  on  $T$  is  $p$ . The set  $\psi(N_d(H))$  is an  $H$ -finite subset of  $\mathcal{G}$  and so belong to  $N_{d'}(T_p)$  for some  $d' = d'(d, p)$  (see also the proof of Corollary 5.5). Since the unique limit point of the set  $N_{d'}(T_p)$  is also  $p$  the map  $\psi$  extends identically to the set  $\mathcal{P}$ . The second statement is similar.  $\square$

**Lemma 7.4.** *For every  $l > 0$  there exists  $\lambda_0 > 1$  such that for any  $\lambda \in ]1, \lambda_0[$  and  $\lambda$ -slow function  $f$  satisfying (1-2) one has: if  $\gamma \subset \Gamma$  is  $\delta_f$ -geodesic then the curve  $\psi(\gamma) \subset \mathcal{G}$  is  $(l, c)$ -tight where  $c$  is the quasi-isometry constant of  $\psi$ .*

*Proof:* For a fixed  $l > 0$  by Lemma 7.2 (applied to  $r = l$ ) there exists  $\lambda_0 > 1$  such that for any  $\lambda \in ]1, \lambda_0[$  and any  $\lambda$ -slow function  $f$ , every part of  $\gamma$  of length less than  $l$  is geodesic in  $\Gamma$ . Then  $\beta = \psi(\gamma)$  is  $c$ -quasigeodesic on every interval of length at most  $l$ . So the first condition of Definition 6.1 is satisfied for  $\beta \subset \mathcal{G}$ .

To prove 6.1.2 assume that

$$|\beta(J)| > l, \quad (**)$$

If first  $\text{diam}(\partial\gamma(J)) \leq l$  then again by Lemma 7.2  $\gamma|_J$  is geodesic in  $\Gamma$ . So  $\beta|_J$  is  $c$ -quasigeodesic in  $\mathcal{G}$ . It follows from (\*\*) that  $\text{diam}(\partial(\beta(J))) > c^{-1}(l) = l/c - c$

If now  $\text{diam}(\partial\gamma(J)) > l$  then we have  $|\partial\beta(J)| > c^{-1}(l)$  since  $\psi$  is a  $c$ -quasi-isometry. The Lemma is proved.  $\square$

Note that the proof of Lemma 7.4 does not use the horospheres to prove the tightness condition 6.1.2. The needed property holds for any part of  $\beta$  of length bigger than  $l$ . The following Corollary shows that it remains valid for a curve in  $\Gamma$  close in the Floyd metric to a Floyd geodesic if the latter one does not belong to the graph.

**Corollary 7.5.** *For every  $l > 0$  there exists  $\lambda_0 > 1$  such that for every  $\lambda \in ]1, \lambda_0[$  and  $\lambda$ -slow function  $f$  if the Floyd geodesic  $\gamma[x, y] \subset \Gamma_f$  joining two distinct points  $x$  and  $y$  does not belong to  $\Gamma$ , then there exists a curve  $\tilde{\gamma}[x, y] \subset \Gamma$  between  $x$  and  $y$  such that  $|L_f(\tilde{\gamma}) - L_f(\gamma)| \leq \varepsilon$  and every part of  $\tilde{\gamma}$  of length  $l$  is  $d$ -geodesic.*

*Furthermore the curve  $\psi(\tilde{\gamma}) \subset \mathcal{G}$  is  $(l, c)$ -tight for the quasi-isometry constant  $c$ .*

*Proof:* For a fixed  $l$  we choose  $\lambda$ -slow function  $f$  such that  $\lambda \geq \frac{l}{l+1}$ . Suppose that a Floyd geodesic  $\gamma[x, y]$  intersects the Floyd boundary  $\partial_f \Gamma$ . Then for any  $\varepsilon > 0$  there exists a curve  $\hat{\gamma} : I \rightarrow \Gamma$  such that  $\hat{\gamma}(\partial I) = \{x, y\}$  and  $|L_f(\hat{\gamma}) - L_f(\gamma)| < \varepsilon$ . Let  $x'$  and  $y'$  be two points on  $\hat{\gamma}$  such that  $d(x', y') = l$ . If the part  $\hat{\gamma}[x', y']$  of  $\hat{\gamma}$  between them is not  $d$ -geodesic we replace it by a  $d$ -geodesic  $\omega = \omega[x', y']$  between  $x'$  and  $y'$ . Then the  $d$ -length of the obtained curve  $\tilde{\gamma}$  is strictly less than that of  $\hat{\gamma}$ . Furthermore by Lemma 7.2 (applied to  $r = l$ ) the curve  $\omega$  is also a Floyd geodesic. So we have

$$L_f(\gamma) \leq L_f(\tilde{\gamma}) \leq L_f(\hat{\gamma}) \leq L_f(\gamma) + \varepsilon.$$

Repeating this procedure with every pair of points of  $\tilde{\gamma}$  situated at the distance  $l$  we strictly decrease its  $d$ -length. Since  $d(x, y) \in \mathbb{Z}_{>0}$  after finitely many steps we obtain a curve (still denoted by  $\tilde{\gamma}$ ) satisfying the first statement.

Since  $\psi : \Gamma \rightarrow \mathcal{G}$  is a  $c$ -quasi-isometry the last part follows from the argument of Lemma 7.4.  $\square$

*Proof of Theorem C.* The group  $G$  acts 3-discontinuously and 2-cocompactly on a compactum  $T$ . Let  $\Gamma$  be a locally finite, connected graph admitting cocompact discontinuous action of  $G$ .

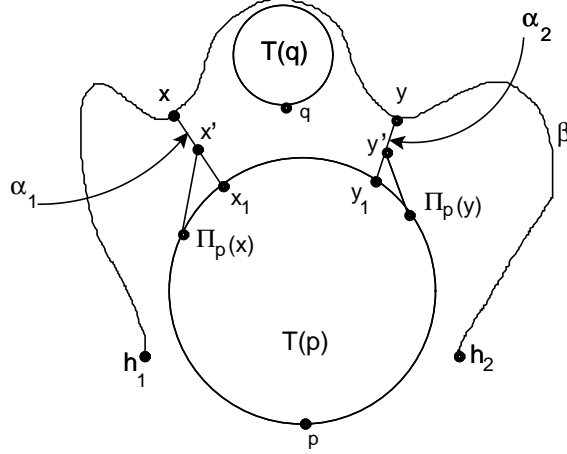
Let  $l_0$  and  $\lambda_0$  be the constants given by Theorem B and Lemma 7.4 (or Corollary 7.5). Let  $f$  be a  $\lambda$ -slow function for  $\lambda \in ]1, \lambda_0[$ . Suppose that  $\gamma = \gamma(h_1, h_2) \subset \Gamma$  is a  $\delta_f$ -geodesic between two elements  $h_1$  and  $h_2$  in the parabolic subgroup  $H$ . Then by Lemma 7.4 the curve  $\beta = \psi(\gamma)$  is  $(l, c)$ -tight in  $\mathcal{G}$ .

A segment of a curve  $\beta \subset \mathcal{G}$  having the extremities at points  $\mathbf{h}_i \in \mathcal{G}$  ( $i = 1, 2$ ) we denote by  $\beta[\mathbf{h}_1, \mathbf{h}_2]$ . By Lemma 7.3 for every  $d > 0$  and  $p \in \mathcal{P}$  there exists  $d' = d'(d, p)$  such that the set  $\psi^{-1}(N_d(T_p))$  belongs to  $N_{d'}(H)$ . So Theorem C follows from the following.

**Proposition 7.6.** *For every  $c > 0$  there exist positive constants  $s$ ,  $d$  and  $l_0$  such that for all  $l > l_0$  every  $(l, c)$ -tight curve  $\beta[\mathbf{h}_1, \mathbf{h}_2] \subset \mathcal{G}$  with  $\mathbf{h}_i \in N_d(T(p))$  ( $i = 1, 2$ ) is situated in  $N_s(T(p))$  for some  $p \in \mathcal{P}$ .*

*Proof of the Proposition.* Since  $\mathcal{P}$  is  $G$ -finite it is enough to prove the statement for a fixed  $p \in \mathcal{P}$ . Suppose that  $\beta$  is a  $(l, c)$ -tight curve where  $l > l_0$  and the constants  $l_0$  and  $c$  are given by Theorem B. So there exists a  $c'$ -quasigeodesic  $\alpha \subset \mathcal{G}$  such that every non-horospherical point  $\mathbf{v}$  of  $\beta$  belongs to the  $w_0$ -neighborhood  $N_{w_0}(\alpha)$  with respect to the distance  $d_A$ . By Lemma 5.4.1 we have  $\forall i \in I : d_A(\alpha(i), T(p)) \leq \text{const}$ . Thus there exists a constant  $R = R(d) > 0$  such that for any non-horospherical point  $\mathbf{v} \in \beta$  we have  $d_A(\mathbf{v}, T(p)) \leq R$ .

Let now  $\beta[\mathbf{x}, \mathbf{y}]$  be a  $d$ -horospherical part of  $\beta$  lying in  $N_d(T(q))$  of another parabolic point  $q$ . Up to increasing the above part of  $\beta$  we can suppose that both extremal points  $\mathbf{x}$  and  $\mathbf{y}$  are non-horospherical. So we have  $d_A(\mathbf{x}, T(p)) \leq R$  and  $d_A(\mathbf{y}, T(p)) \leq R$ . Let  $\mathbf{x}_1$  and  $\mathbf{y}_1$  be points on  $T(p)$  realizing these distances respectively. Denote by  $\alpha_1 = [\mathbf{x}, \mathbf{x}_1]$  and  $\alpha_2 = [\mathbf{y}, \mathbf{y}_1]$  the corresponding geodesics (see Figure below).



Let  $\Pi_p(\mathbf{x})$  and  $\Pi_p(\mathbf{y})$  be the projections of  $\mathbf{x}$  and  $\mathbf{y}$  on  $T(p)$ . By Lemma 5.3.1 we have  $d_A(\alpha_1, \Pi_p(\mathbf{x})) = d_A(\mathbf{x}', \Pi_p(\mathbf{x})) \leq L$  for some constant  $L$  depending only on  $R$ , where  $\mathbf{x}' \in \alpha_1$ . Hence  $d_A(\mathbf{x}, \Pi_p(\mathbf{x})) \leq R + L$  and similarly  $d_A(\mathbf{y}, \Pi_p(\mathbf{y})) \leq R + L$ . By Proposition 3.32.2 the set  $\Pi_p(T(q))$  is finite and so is  $\Pi_p(N_d(T(q)))$ . So there exists a constant  $C > 0$  such that  $d_A(\Pi_p \mathbf{x}, \Pi_p \mathbf{y}) \leq C$ . Therefore  $d_A(\mathbf{x}, \mathbf{y}) \leq C + 2R + 2L$ . The above constants  $C$ ,  $R$  and  $L$  depend only on  $p$  so we can choose the parameter  $l$  from Theorem B satisfying  $l > \max(l_0, C + 2R + 2L)$ . Then the segment  $\beta[\mathbf{x}, \mathbf{y}]$  is  $c$ -quasigeodesic whose length is bounded by  $c(C + 2R + 2L) + c$ . Hence  $\beta[\mathbf{x}, \mathbf{y}] \subset N_s(T(p))$  where  $s = R + c(C + 2R + 2L) + c$ . Theorem C is proved.  $\square$

Since every parabolic subgroup  $H$  is quasiconvex in  $G$  there exists a quasi-isometric map  $\varphi$  of the group  $H$  into the graph  $\Gamma$ . We have the following.

**Corollary 7.7.** *For the constant  $\lambda_0$  from Theorem C and every  $\lambda \in ]1, \lambda_0[$  let  $f$  be a  $\lambda$ -slow Floyd function satisfying in addition the following assumption:*

$$\frac{f(n)}{f(2n)} \leq \kappa \quad (n \in \mathbb{N}) \quad (3)$$

*for some constant  $\kappa > 0$ . Let  $p$  be a parabolic point for the action of  $G$  on  $T$  and  $H = \text{Stab}_G p$  be its stabilizer. Then  $\varphi$  extends injectively to the Floyd boundaries:*

$$\varphi : \overline{H}_f \rightarrow \overline{\Gamma}_f. \quad (4)$$

$\square$

**Remark.** Note that every polynomial type function  $f(n) = (n + 1)^{-k}$  ( $k > 1$ ) satisfies the conditions (1-3) for any fixed  $\lambda > 1$  and  $\kappa > 0$  ( $n > n_0$ ).

*Proof of Corollary 7.7.* We suppose that  $H \subset \Gamma^0$  and  $\varphi : H \hookrightarrow \Gamma^0$  is the identity map inducing the quasi-isometry between the word metrics. Let  $d'(\cdot, \cdot)$  and  $d(\cdot, \cdot)$  be the graph distances of  $H$  and  $\Gamma$  respectively. We also denote by  $\delta_{f,H}$  and  $\delta_{f,G}$  the corresponding Floyd distances with respect to a fixed basepoint  $v \in H$ . Since  $f$  satisfies (3) by [GePo1, Lemma 2.5] the map  $\varphi$  extends to a

Lipschitz map (denoted by the same letter)  $\varphi : \overline{H}_f \rightarrow \overline{\Gamma}_f$  between the Floyd completions of  $H$  and  $\Gamma$ .

Let  $x, y \in H \subset \Gamma$  be two distinct points. If the Floyd geodesic between  $x$  and  $y$  belongs to  $\Gamma$  we denote it by  $\gamma$ ; if not for any  $\varepsilon \in ]0, 1[$  let  $\gamma$  be the  $(l, c)$ -tight curve ( $l > l_0$ ) given by Corollary 7.5 whose Floyd length is  $\varepsilon$ -close to that of the Floyd geodesic. In the first case by Theorem C there exists a constant  $R = R(H)$  such that  $\gamma \subset N_R(H)$ , and in the second case the same conclusion for the curve  $\gamma$  follows from Proposition 7.6.

We have  $L_f(\gamma) = \sum_{i=1}^l f(d(v, \{x_i, x_{i+1}\}))$ . Denote by  $x'_i \in H$  one of the closest vertices to  $x_i$  in  $H$  ( $i = 1, \dots, l$ ). By Theorem C there exists a constant  $R > 0$  such that  $d(x_i, x'_i) \leq R$ . Thus  $d(x'_i, x'_{i+1}) \leq 2R + 1$ . So for any vertex  $x'_{ij}$  on a geodesic in  $H$  between  $x'_i$  and  $x'_{i+1}$  we obtain  $d(v, \{x_i, x_{i+1}\}) \leq (3R + 1) + d(v, x'_{ij})$ . Since  $\varphi$  is quasi-isometric we have  $1/\alpha \cdot d'(v, x'_{ij}) - \beta \leq d(v, x'_{ij}) \leq \alpha d'(v, x'_{ij}) + \beta$  for some constants  $\alpha$  and  $\beta$ . Let  $\gamma' = \gamma'(x, y) \subset H$  be the curve between  $x$  and  $y$  obtained by connecting the vertices  $x'_i$  and  $x'_{i+1}$  by geodesics segments in  $H$  passing through  $x'_{ij}$ . We have  $\alpha \cdot (d(x'_i, x'_{i+1}) + \beta) \cdot f(d(v, \{x_i, x_{i+1}\})) \geq d'(x'_i, x'_{i+1}) \cdot f(d(v, \{x_i, x_{i+1}\})) = \sum_j f(d(v, \{x_i, x_{i+1}\}))$ . Thus  $f(d(v, \{x_i, x_{i+1}\})) \geq \frac{1}{2\alpha R + \beta + \alpha} \sum_j f(\alpha d'(v, x'_{ij}) + m_1)$ , where  $m_1 = \beta + 3R + 1$ . The conditions (1) and (3) yield

$$L_{f,G}(\gamma) \geq \frac{L_{f,H}(\gamma')}{(2\alpha R + \beta + \alpha)\lambda^{m_1 K^{k_1}}} \geq \frac{\delta_{f,H}(x, y)}{(2\alpha R + \beta + \alpha)\lambda^{m_1 K^{k_1}}}, \quad (5)$$

where  $k_1 = \min\{k : 2^k > \alpha\}$ . Since for every  $\varepsilon \in ]0, 1[$  there exists a curve  $\gamma$  satisfying (5) and for which  $L_{f,G}(\gamma) \leq \delta_{f,G}(x, y) + \varepsilon$  we have

$$\forall x, y \in H \quad \delta_{f,G}(x, y) \geq \frac{1}{(2\alpha R + \beta + \alpha)\lambda^{m_1 K^{k_1}}} \cdot \delta_{f,H}(x, y). \quad (6)$$

By continuity the inequality (6) remains valid for every pair of distinct points  $x, y \in \overline{H}_f$ . So the map  $\varphi : \overline{H}_f \rightarrow \overline{\Gamma}_f$  is injective. The Corollary is proved.  $\square$

If  $G$  acts on  $T$  is 3-discontinuously and 2-cocompactly then the kernel of the equivariant Floyd map  $F$  from the Floyd boundary  $\partial_f G$  of the Cayley graph of  $G$  to  $T$  is described in [GePo1, Thm A]. Namely if it is not a single point then it is equal to the topological boundary  $\partial(\text{Stab}_G p)$  of the stabilizer  $\text{Stab}_G p$  of a parabolic point  $p \in T$ . We denote by  $\partial_f \text{Stab}_G p$  the Floyd boundary of  $\text{Stab}_G p$  corresponding to the function  $f$ . By Corollary 7.7 we have that  $\varphi|_{\partial_f H}$  is a homeomorphism. So the following is immediate.

**Corollary 7.8.** *For every  $\lambda \in ]1, \lambda_0[$  and each  $\lambda$ -slow function  $f$  satisfying (1 – 3) one has*

$$F^{-1}(p) = \partial_f(\text{Stab}_G p), \quad (7)$$

*for every parabolic point  $p \in T$ .*

$\square$

Corollary 7.8 answers positively our question [GePo1, 1.1] and provides complete generalization of the theorem of Floyd [F] for the relatively hyperbolic groups.

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